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Hartree-Fock type systems: Existence of ground states and asymptotic behavior

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Abstract

In this paper we consider a Hartree-Fock type system made by two Schrödinger equations in presence of a Coulomb interacting term and a *cooperative* pure power and subcritical nonlinearity, driven by a suitable parameter $\beta \geq 0$. We show the existence of semitrivial and vectorial ground states solutions depending on the parameters involved. The asymptotic behavior with respect to the parameter β of these solutions is also studied.

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Contents

1. Introduction 581
Notations 586
2. Preliminary results 586

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3. Existence and nonexistence results . . . 590
3.1. A Pohozaev identity and the nonexistence result . . . 590
3.2. Existence of a radial ground state . . . 592
4. The case $\beta = 0$ . . . 600
5. The case $\beta > 0$ and small . . . 602
6. The case β large . . . 609
7. The particular case $\beta = 2^{q-1} - 1$ and $q \in [2, 3)$ . . . 610
Data availability . . . 611
Acknowledgments . . . 611
Appendix A. Proof of Lemma 2.4 . . . 611
References . . . 613

1. Introduction

In the study of a molecular system made of M nuclei interacting via the Coulomb potential with N electrons, the starting point is the $(M+N)$-body Schrödinger equation

$$i\hbar\partial_t \Psi = -\frac{\hbar^2}{2}\sum_{j=1}^{M+N}\frac{1}{m_j}\Delta_{x_j}\Psi + \frac{e^2}{8\pi\varepsilon_0}\sum_{\substack{j,k=1\\ j\neq k}}^{M+N}\frac{Z_j Z_k}{|x_j - x_k|}\Psi, \qquad \Psi : \mathbb{R}\times\mathbb{R}^{3(M+N)} \to \mathbb{C}$$

where the constants eZ_j's are the charges and in particular the charge numbers Z_j's are positive for the nuclei and -1 for the electrons.

Its complexity led to consider various approximations to describe the stationary states with simpler models.

A possible approximation, used in particular in models of Quantum Chemistry, is the Born-Oppenheimer approximation. Here the nuclei are considered as classical point particles and a fundamental assumption is that they are much heavier than electrons (see e.g. [5] for a mathematical treatment).

Starting from the Born-Oppenheimer model, a further possible approximation is the Hartree-Fock method, which is generally considered fundamental to much of electronic structure theory and represents the basis of molecular orbital theory. It is variational and the electrons are considered as occupying single-particle orbitals making up the wavefunction. Each electron feels the presence of the other electrons indirectly through an effective potential. Thus, each orbital is affected by the presence of electrons in other orbitals.

This was introduced by Hartree in [15] through the use of some particular test functions, without taking into account the Pauli principle. Subsequently, Fock in [12] and Slater in [30], to take into account the Pauli principle, chose a different class of test functions, the Slater determinants, obtaining a system of N coupled nonlinear Schrödinger equations

$$-\frac{\hbar^2}{2m}\Delta\psi_k + V_{\rm ext}\psi_k + \Big(\int_{\mathbb{R}^3}|x-y|^{-1}\sum_{j=1}^{N}|\psi_j(y)|^2dy\Big)\psi_k + (V_{\rm ex}\psi)_k = E_k\psi_k, \quad k = 1,\dots,N,$$

where $\psi_k : \mathbb{R}^3 \to \mathbb{C}$, $V_{\rm ext}$ is a given external potential,

$$(V_{\rm ex}\psi)_k := -\sum_{j=1}^{N}\psi_j \int\limits_{\mathbb{R}^3} \frac{\psi_k(y)\overline{\psi}_j(y)}{|x-y|}dy$$

is the k'th component of the *crucial exchange term*, and E_k is the k'th eigenvalue.
A further relevant approximation for the exchange potential $V_{\rm ex}\psi$ is due to Slater in [31] (see also Dirac in [10] in a different context), namely

$$(V_{\rm ex}\psi)_k \approx -C\Big(\sum_{j=1}^{N}|\psi_j|^2\Big)^{1/3}\psi_k. \tag{1.1}$$

Moreover, slightly different local approximations have been done in [14,16]. For further models we refer to [27] and references therein.
We emphasize that in these last approximations there is a strong dependence on the electron density function $\sum_{j=1}^{N}|\psi_j|^2$.
For more details about the Hartree-Fock method we refer the reader to [4,9,13,23,24,27,34] and references therein, and, for a mathematical approach to [18,20,35].

In this paper we take $N=2$ and we assume

$$\begin{aligned}(V_{\rm ex}\psi) &= -C\begin{pmatrix}|\psi_1|^{q-2}\psi_1 & \beta|\psi_1|^{q-2}\psi_1\\ \beta|\psi_2|^{q-2}\psi_2 & |\psi_2|^{q-2}\psi_2\end{pmatrix}\begin{pmatrix}|\psi_1|^q\\ |\psi_2|^q\end{pmatrix}\\ &= -C\begin{pmatrix}|\psi_1|^{2q-2}\psi_1+\beta|\psi_1|^{q-2}|\psi_2|^q\psi_1\\ |\psi_2|^{2q-2}\psi_2+\beta|\psi_1|^q|\psi_2|^{q-2}\psi_2\end{pmatrix}\end{aligned} \tag{1.2}$$

where q,β are suitable parameters.
Observe that, for $q=2$, the approximation in (1.2) becomes

$$(V_{\rm ex}\psi) = -C\begin{pmatrix}\psi_1 & \beta\psi_1\\ \beta\psi_2 & \psi_2\end{pmatrix}\begin{pmatrix}|\psi_1|^2\\ |\psi_2|^2\end{pmatrix} = -C\begin{pmatrix}(|\psi_1|^2+\beta|\psi_2|^2)\psi_1\\ (\beta|\psi_1|^2+|\psi_2|^2)\psi_2\end{pmatrix},$$

that is similar to the one applied by Slater in (1.1), with a different power of the electron density function which is also *perturbed* by the parameter β.
Considering ψ_1 and ψ_2 real functions, renaming them as u, v, and taking, for simplicity, $C=1$, we get

$$\begin{cases}-\Delta u+u+\lambda\phi_{u,v}u=|u|^{2q-2}u+\beta|v|^q|u|^{q-2}u\\ -\Delta v+v+\lambda\phi_{u,v}v=|v|^{2q-2}v+\beta|u|^q|v|^{q-2}v\end{cases} \text{in } \mathbb{R}^3, \tag{$\mathcal{S}_{\lambda,\beta}$}$$

where

$$\phi_{u,v}(x) := \int\limits_{\mathbb{R}^3}\frac{u^2(y)+v^2(y)}{|x-y|}dy \in D^{1,2}(\mathbb{R}^3),$$

where this last space is the closure of the test functions in the L^2-norm of the gradient.
Observe that $\phi_{u,v}$ is the unique solution of

$$-\Delta\phi = 4\pi(u^2+v^2) \quad \text{in } \mathbb{R}^3.$$

Thus, system ($\mathcal{S}_{\lambda,\beta}$) can be also seen as a Schrödinger-Poisson type system (see e.g. [11]).

A particular case of system ($\mathcal{S}_{\lambda,\beta}$), when $\lambda = 0$, leads to the local weakly coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + u = |u|^{2q-2}u + \beta|v|^q|u|^{q-2}u \\ -\Delta v + \omega^2 v = |v|^{2q-2}v + \beta|u|^q|v|^{q-2}v \end{cases} \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

for $0 < \omega^2 \leq 1$, which has been intensively studied in the past fifteen years. Applying variational methods, the first works are authored by Lin and Wei [19] and also by Ambrosetti and Colorado [1], Maia, Montefusco, and Pellacci [21], Bartsch and Wang [2], Sirakov [29], then followed by an extensive literature presenting investigations of different aspects and variations of this problem.
In fact this system is obtained when looking for solitary wave solutions of two coupled nonlinear Schrödinger equations which model, for instance, binary mixtures of Bose-Einstein condensates or propagation of wave packets in nonlinear optics. In the present scenario, the self-interaction is attractive (self-focusing) and the interaction between the two components may be either attractive ($\beta > 0$) or repulsive ($\beta < 0$). Many different and clever approaches have been provided in order to find ranges of parameter β for which a positive (ground state) solution (u, v) of the system is vectorial (namely having both nontrivial components) and so distinguish them from the semitrivial ones $(u, 0)$ and $(0, v)$. So far a remarkable amount of information has been made available on this matter, including the proof in [22] of a threshold $\beta(\omega, q, n)$ for existence or nonexistence of vector ground states for problem (1.3) in $\mathbb{R}^n$.

The system above also arises as population dynamics are modeled and their associated reaction-diffusion equations in bounded or unbounded domains are studied using variational techniques; among many interesting works on this matter there are [6,7,32] and references therein. When, for instance, an analysis is performed of the limiting case with respect to a parameter β which describes interspecies competitions, going to plus or minus infinity, possible segregation states of two or more competing species are identified, leading to configurations where the populations occupy disjoint habitats.

In this paper we study the existence of solutions to problem ($\mathcal{S}_{\lambda,\beta}$) in the unknowns $(u, v) \in \mathrm{H} := H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. In particular we are interested in nontrivial solutions, namely $(u, v) \in \mathrm{H} \setminus \{0\} := \mathrm{H} \setminus \{(0, 0)\}$.

Our approach in solving problem ($\mathcal{S}_{\lambda,\beta}$) is variational. Indeed a C^1 energy functional in H can be defined such that its critical points give exactly the solutions of our system.

However in order to deal with compactness issues, we will work (except for the nonexistence result) in the radial setting and we will use the compact embedding of $H^1_{\mathrm{r}}(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ for $p \in (2, 6)$, see e.g. [3,33]. Then the functional will be restricted to $\mathrm{H_r} := H^1_{\mathrm{r}}(\mathbb{R}^3) \times H^1_{\mathrm{r}}(\mathbb{R}^3)$ and the solutions will be found in $\mathrm{H_r}$. The invariance of the functional under rotations and the Palais' Principle of Symmetric Criticality [26] makes natural this constraint.

Actually we are interested in the existence of *ground state solutions*: with these terms we mean radial solutions whose energy is minimal among all the other radial ones.

Such *definition* is motivated by the fact that for our system ($\mathcal{S}_{\lambda,\beta}$), as well as for the corresponding scalar problem

$$-\Delta u + u + \lambda\phi_u u = |u|^{2q-2}u \quad \text{in } \mathbb{R}^3, \quad \phi_u(x) := \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy, \tag{1.4}$$

the classical Schwarz symmetrization or the polarization arguments (see [17,25]), that are enough to treat the nonlocal term, and so to prove the radial symmetry of the ground state solutions for the Choquard equation, are (or seem to be, respectively) not sufficient to guarantee the radial symmetry of our ground states. Indeed, in our case, as observed by Lieb in [17], the Riesz inequality implies that the energy increases when we pass to the symmetrized function.

In order to state our main result concerning the existence of ground state solutions for $q \in (3/2, 3)$, their vectorial or semitrivial nature, and their asymptotic behavior with respect to the parameter β, let us first recall that in [28] it was proved that, for any $\lambda > 0$, the equation (1.4) possesses a radial ground state solution among all the radial solutions which will be denoted hereafter with $\mathfrak{w} \in H^1_{\rm r}(\mathbb{R}^3)$.

Observe that, whenever a ground state of ($\mathcal{S}_{\lambda,\beta}$) is semitrivial, then, necessarily, it is of the type $(\mathfrak{w}, 0)$ or $(0, \mathfrak{w})$.

We have

Theorem 1.1. *Let $q \in (3/2, 3)$, $\lambda > 0$, and $\beta \geq 0$. Then ($\mathcal{S}_{\lambda,\beta}$) has a radial ground state solution $(\mathfrak{u}_\beta, \mathfrak{v}_\beta) \neq (0, 0)$. Moreover:*

(i) *if $\beta = 0$, the ground state solution is semitrivial;*
(ii) *if $\beta \in (0, 2^{q-1} - 1)$ and $q \in [2, 3)$, the ground state solution is semitrivial;*
(iii) *if $\beta \in (0, q - 1)$ and $q \in (3/2, 2)$, the ground state solution is vectorial and*

$$\lim_{\beta\to 0^+} \mathrm{dist}_{\mathrm{H}}(\mathcal{G}_\beta, \mathcal{G}_0) = 0$$

where $\mathcal{G}_\beta := \{(\mathfrak{u}_\beta, \mathfrak{v}_\beta) \in \mathrm{H}_{\rm r} : (\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ is a ground state of ($\mathcal{S}_{\lambda,\beta}$)$\}$ and $\mathcal{G}_0 := \{(\mathfrak{w}, 0), (0, \mathfrak{w})\}$;

(iv) *if*

$$\beta \in \begin{cases} [q-1, +\infty) & \text{for } q \in (3/2, 2), \\ (2^{q-1} - 1, +\infty) & \text{for } q \in [2, 3), \end{cases} \tag{1.5}$$

the ground state $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ is vectorial and

$$\lim_{\beta\to+\infty} (\mathfrak{u}_\beta, \mathfrak{v}_\beta) = (0, 0) \text{ in } \mathrm{H}_{\rm r}; \tag{1.6}$$

(v) *if $\beta = 2^{q-1} - 1$ and $q \in [2, 3)$, system ($\mathcal{S}_{\lambda,\beta}$) admits both semitrivial and vectorial ground states.*

Some remarks on our result are now in order.

The presence of the nonlocal Coulomb type coupling in ($\mathcal{S}_{\lambda,\beta}$) implies several difficulties with respect to system like (1.3), in particular for what concerns the semitrivial or vectorial nature of

the ground states, which is, actually, the main goal of the paper.
Indeed system (1.3) when $\beta = 0$ or when we consider semitrivial solutions, reduces to single equation

$$-\Delta u + u = |u|^{2q-2}u \quad \text{in } \mathbb{R}^3.$$

For such equation, well known results have been obtained about uniqueness of the positive solution, its nondegeneracy, its radial symmetry and exponential decay. These facts are used in the study of (1.3) (see [21,22]).
In our case, even for $\beta = 0$ the system remains coupled in the nonlocal term.
Moreover, even if, for semitrivial solutions, system ($\mathcal{S}_{\lambda,\beta}$) reduces to a single equation, for such equation no result about uniqueness, nondegeneracy, and eventual symmetries of positive solution is known.
Finally, to deal with powers $q \in (3/2, 3)$, following [28], we use a rescaling (see (2.11)) which generates different behaviors of the terms in the functionals but that anyway allows us to project any nontrivial couple (u, v) in a suitable manifold. Actually, for the simpler case $q \in (2, 3)$, the usual projection on the Nehari manifold is enough.
Nevertheless, our analysis shows that the nature of the ground states depends on the local nonlinearity. Indeed our results are comparable with the ones in [22], even if they are obtained in a different way: we start from the existence of ground states and, using the maximum values of a suitable one variable function related to the local nonlinearity (see Lemma 2.4), we estimate the ground state energy level and construct also a particular family of ground states (see Lemma 3.7), that, in the particular case $q = 2$ and $\beta = 1$, gives infinitely many ground states.

Additionally, due to the symmetry in u and v of ($\mathcal{S}_{\lambda,\beta}$), it is easy to obtain nontrivial solutions with $u = v$ (see Remark 2.2). For β large enough, such solutions are ground states (Theorem 6.1) and, for β small, they are not (Theorem 5.6).

Finally, the solutions we find are classical. Indeed, if $(u, v) \in \mathrm{H_r}$, then $\phi_{u,v} \in W^{2,3}_{\rm loc}(\mathbb{R}^3)$ and then it is $C^{0,\alpha}_{\rm loc}(\mathbb{R}^3)$. But then by bootstrap arguments $u, v \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^3)$ which in turn implies $\phi_{u,v} \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^3)$. Moreover, by the Maximum Principle, every nontrivial component of a solution can be assumed strictly positive without loss of generality.

Of course our problem can be written using the equivalent complex notation $\psi := u + iv$. Observe that, with such a notation,

$$\int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} dy = \int_{\mathbb{R}^3} \frac{|\psi(y)|^2}{|x-y|} dy,$$

depending only on $|\psi|$. For our scopes, especially in order to distinguish between semitrivial and vectorial ground states, in the analysis it should be necessary to use real and imaginary parts of ψ and so we will proceed using the vectorial notation (u, v).

Additionally, we prove also the following nonexistence result.

Theorem 1.2. *In* $\mathrm{H} \cap (L^{2q}(\mathbb{R}^3) \times L^{2q}(\mathbb{R}^3)) \cap (L^\infty_{\rm loc}(\mathbb{R}^3) \times L^\infty_{\rm loc}(\mathbb{R}^3))$, *system* ($\mathcal{S}_{\lambda,\beta}$) *has only the trivial solution if* $q \geq 3$ *and no solution with* fixed sign *if* $q \in [1/2, 1]$.

Here, with *fixed sign solution*, we mean couples (u, v) where each component is strictly positive or negative.

The paper is organized as follows.
In Section 2 we present few preliminaries in order to prove our results. In particular we recall some results in [28] that will be used to compare the ground state level of our functional (for example to study the asymptotic behavior). We give also the variational setting for our problem.
In Section 3 we prove the nonexistence result, Theorem 1.2, which is based on a Pohozaev identity associated to the problem. Then we give also the proof of the existence of a nontrivial ground state in Theorem 1.1.
Then item (i) is proved in Section 4, (ii) and (iii) are proved in Section 5, (iv) is proved in Section 6, and (v) in Section 7.
We complete Section 5 and Section 6 showing that some particular solutions arising from the study of the single equation (see Remark 2.2) are or not ground states (see Theorem 5.6 and Theorem 6.1, respectively).

Notations

- Unless otherwise stated, integrals will always be considered on the whole $\mathbb{R}^3$ with the Lebesgue measure.
- We denote with $\|\cdot\|$ the norm in $H^1(\mathbb{R}^3)$ and with $\|\cdot\|_p$ the standard L^p- norm.
- We denote with ε_n a generic sequence which vanishes as n tends to infinity and with C a suitable positive constant that can vary from line to line.

Other notations will be introduced whenever needed.

2. Preliminary results

In order to prove our results, let us first recall some facts about (1.4). In [28] it was proved that for any $\lambda > 0$ and $q \in (3/2, 3)$, equation (1.4) has a *radial ground state* solution $\mathfrak{w} \in H^1_{\rm r}(\mathbb{R}^3) \setminus \{0\}$. It is found as a minimizer of the C^1-functional

$$\mathcal{I}_{\lambda,0}(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{\lambda}{4}\int u^2\phi_u - \frac{1}{2q}\|u\|_{2q}^{2q}, \quad u \in H^1_{\rm r}(\mathbb{R}^3)$$

on the constraint

$$\mathcal{N}^\lambda := \left\{u \in H^1_{\rm r}(\mathbb{R}^3) : \mathcal{J}_{\lambda,0}(u) = 0\right\}, \tag{2.1}$$

where

$$\mathcal{J}_{\lambda,0}(u) := \frac{3}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{3}{4}\lambda\int \phi_u u^2 - \frac{4q-3}{2q}\|u\|_{2q}^{2q}.$$

The set $\mathcal{N}^\lambda$ is obtained as a *linear combination* of the Nehari identity

$$\|\nabla u\|_2^2 + \|u\|_2^2 + \lambda\int \phi_u u^2 - \|u\|_{2q}^{2q} = 0$$

and the Pohozaev identity

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{3}{2}\|u\|_2^2 + \frac{5}{4}\lambda\int \phi_u u^2 - \frac{3}{2q}\|u\|_{2q}^{2q} = 0.$$

Given $u \not\equiv 0$, consider the path

$$\zeta_u(t) := t^2 u(t\cdot), \quad t \geq 0. \tag{2.2}$$

Note that

$$\begin{aligned}\mathcal{I}_{\lambda,0}(\zeta_u(t)) &= \frac{t^3}{2}\|\nabla u\|_2^2 + \frac{t}{2}\|u\|_2^2 + \frac{\lambda}{4}t^3\int u^2\phi_u - \frac{t^{4q-3}}{2q}\|u\|_{2q}^{2q},\\ \mathcal{J}_{\lambda,0}(\zeta_u(t)) &= \frac{3}{2}t^3\|\nabla u\|_2^2 + \frac{1}{2}t\|u\|_2^2 + \frac{3}{4}\lambda t^3\int \phi_u u^2 - \frac{4q-3}{2q}t^{4q-3}\|u\|_{2q}^{2q},\end{aligned} \tag{2.3}$$

and $t \mapsto \mathcal{I}_{\lambda,0}(\zeta_u(t))$ has a unique critical point, denoted with $t_u > 0$ corresponding to its maximum. The elements of $\mathcal{N}^\lambda$ are then all of type $\zeta_u(t_u)$ due to the fact that

$$\mathcal{J}_{\lambda,0}(\zeta_u(t)) = \frac{d}{dt}\mathcal{I}_{\lambda,0}(\zeta_u(t)).$$

In particular $u \in \mathcal{N}^\lambda$ if and only if $t_u = 1$ and then

$$0 < \mathcal{I}_{\lambda,0}(\mathfrak{w}) = \inf_{u\in\mathcal{N}^\lambda}\mathcal{I}_{\lambda,0}(u) = \inf_{u\in H^1_{\mathrm{r}}(\mathbb{R}^3)\setminus\{0\}}\mathcal{I}_{\lambda,0}(\zeta_u(t_u)) = \inf_{u\in H^1_{\mathrm{r}}(\mathbb{R}^3)\setminus\{0\}}\max_{t>0}\mathcal{I}_{\lambda,0}(\zeta_u(t)). \tag{2.4}$$

Remark 2.1. Of course $(\mathfrak{w}, 0)$ and $(0, \mathfrak{w})$ are semitrivial solutions of our system ($\mathcal{S}_{\lambda,\beta}$) for any β and so, since $I_{\lambda,\beta}(u, 0) = I_{\lambda,\beta}(0, u) = \mathcal{I}_{\lambda,0}(u)$, they are necessarily ground state whenever the ground state is semitrivial.

For future reference we set

$$\mathfrak{n} := \mathcal{I}_{\lambda,0}(\mathfrak{w}).$$

Moreover, the same arguments of [28] can be repeated for the equation

$$-\Delta u + u + 2\lambda\phi_u u = (1+\beta)|u|^{2q-2}u \quad \text{in } \mathbb{R}^3 \tag{2.5}$$

where $\beta \geq 0$, leading to the existence of a ground state solution $\mathfrak{z}_\beta$ that minimizes the functional

$$\mathcal{I}_{2\lambda,\beta}(u) := \mathcal{I}_{2\lambda,0}(u) - \frac{\beta}{2q}\|u\|_{2q}^{2q} = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{\lambda}{2}\int \phi_u u^2 - \frac{1+\beta}{2q}\|u\|_{2q}^{2q}$$

on the set of $u \in H^1_{\mathrm{r}}(\mathbb{R}^3)$ satisfying

$$\mathcal{J}_{2\lambda,\beta}(u) := \frac{3}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{3}{2}\lambda\int \phi_u u^2 - \frac{4q-3}{2q}(1+\beta)\|u\|_{2q}^{2q} = 0.$$

Coming back to our system ($\mathcal{S}_{\lambda,\beta}$), observe that it can be written as

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{2q-2}u + \beta|v|^q|u|^{q-2}u \\ -\Delta v + v + \lambda\phi v = |v|^{2q-2}v + \beta|u|^q|v|^{q-2}v \qquad \text{in } \mathbb{R}^3. \\ -\Delta\phi = 4\pi(u^2+v^2) \end{cases} \tag{2.6}$$

Moreover

$$\int_{\mathbb{R}^3} |\nabla\phi_{u,v}|^2 = 4\pi\int_{\mathbb{R}^3}(u^2+v^2)\phi_{u,v} \tag{2.7}$$

from which the estimate follows

$$\|\nabla\phi_{u,v}\|_2 \le C\left(\|u\|^2+\|v\|^2\right).$$

It is standard to see that the weak solutions of ($\mathcal{S}_{\lambda,\beta}$) are characterized as the critical points of the C^1 functional defined on H

$$I_{\lambda,\beta}(u,v) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{\lambda}{4}\int(u^2+v^2)\phi_{u,v}$$
$$- \frac{1}{2q}(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q}) - \frac{\beta}{q}\int|u|^q|v|^q.$$

Remark 2.2. Observe that, for every $\beta \ge 0$ and $u \in H^1(\mathbb{R}^3)$,

$$I_{\lambda,\beta}(u,u) = 2\mathcal{I}_{2\lambda,\beta}(u) \tag{2.8}$$

and, (u,u) is a solution of ($\mathcal{S}_{\lambda,\beta}$) if and only if u is a solution of (2.5).

If (u,v) is a solution of ($\mathcal{S}_{\lambda,\beta}$), multiplying the first equation of the system by u and the second one by v we see that $(u,v) \in \mathrm{H}$ satisfies the Nehari type identities

$$\|\nabla u\|_2^2 + \|u\|_2^2 + \lambda\int u^2\phi_{u,v} = \|u\|_{2q}^{2q} + \beta\int|u|^q|v|^q, \tag{2.9}$$

$$\|\nabla v\|_2^2 + \|v\|_2^2 + \lambda\int v^2\phi_{u,v} = \|v\|_{2q}^{2q} + \beta\int|u|^q|v|^q. \tag{2.10}$$

Given $(u,v) \in \mathrm{H}\setminus\{0\}$, we denote with $\gamma_{u,v} : [0,+\infty[\to \mathrm{H}$ the curve

$$\gamma_{u,v}(t) := (t^2u(t\cdot), t^2v(t\cdot)). \tag{2.11}$$

By a simple calculation we have that

$$I_{\lambda,\beta}(\gamma_{u,v}(t)) = \frac{t^3}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{t}{2}(\|u\|_2^2 + \|v\|_2^2) + \frac{\lambda}{4}t^3 \int (u^2+v^2)\phi_{u,v}$$
$$- \frac{t^{4q-3}}{2q}\left(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2\beta \int |u|^q|v|^q\right),$$

which will be useful in our arguments.

For future developments, we need the following results.

Lemma 2.3. *Let $\mu, \nu, \sigma > 0$, $p > 3$, and consider the function $\mathfrak{f}_\sigma(t) := \mu t + \nu t^3 - \sigma t^p$. Then*

(a) *$\mathfrak{f}_\sigma$ has a unique critical point $\mathfrak{t}_\sigma > 0$ which corresponds to its maximum and there exists a unique $\mathfrak{T}_\sigma > \mathfrak{t}_\sigma$ such that $\mathfrak{f}_\sigma(\mathfrak{T}_\sigma) = 0$;*
(b) *$\lim_{\sigma\to+\infty} \mathfrak{f}_\sigma(\mathfrak{t}_\sigma) = 0$;*
(c) *$\lim_{\sigma\to+\infty} \mathfrak{T}_\sigma = 0$ and $\mu = \lim_{\sigma\to+\infty} \sigma\mathfrak{T}_\sigma^{p-1}$.*

Proof. Property (a) is essentially [28, Lemma 3.3] and is trivial.
Let us prove (b). Since $p > 3$, then, necessarily, $\mathfrak{t}_\sigma \to 0$ as $\sigma \to +\infty$. Indeed, if there exists $\bar{\mathfrak{t}} > 0$ and a divergent sequence $\{\sigma_n\}$ such that $\mathfrak{t}_{\sigma_n} > \bar{\mathfrak{t}}$, then

$$\mu = p\sigma_n \mathfrak{t}_{\sigma_n}^{p-1} - 3\nu \mathfrak{t}_{\sigma_n}^2 = \mathfrak{t}_{\sigma_n}^2(p\sigma_n \mathfrak{t}_{\sigma_n}^{p-3} - 3\nu) > \bar{\mathfrak{t}}^2(p\sigma_n \bar{\mathfrak{t}}^{p-3} - 3\nu) \to +\infty$$

giving a contradiction. Thus

$$\mathfrak{f}_\sigma(\mathfrak{t}_\sigma) = \mathfrak{t}_\sigma\left(\frac{p-1}{p}\mu + \frac{p-3}{p}\nu t_\sigma^2\right) \to 0 \text{ as } \sigma \to +\infty.$$

As for (c), since $\mathfrak{T}_\sigma$ satisfies

$$\mu = \sigma\mathfrak{T}_\sigma^{p-1} - \nu\mathfrak{T}_\sigma^2 \tag{2.12}$$

we deduce, as in item (b), that $\mathfrak{T}_\sigma \to 0$ as $\sigma \to +\infty$ and so, coming back to (2.12), we conclude. □

Now we state a fundamental tool that will allow us to distinguish the nature of the ground states pairs, identifying whether they are semitrivial or vectorial (see also Remark 3.8). Its proof is quite technical and involves simple analytical arguments. So we postpone it in the Appendix A.

Lemma 2.4. *Let $\mathfrak{h}_\beta(y) := y^q + (1-y)^q + 2\beta y^{q/2}(1-y)^{q/2}$, $y \in [0,1]$, $\beta \geq 0$ and $q > 1$.*

(i) *If $\beta = 0$, then $\mathfrak{h}_0(y) \leq 1$ and the equality holds only in the endpoints $y = 0, 1$.*
(ii) *If $q \in (3/2, 2)$, then, for any fixed $\beta > 0$, there exists a unique $y_\beta \in (0, 1/2]$ such that $\mathfrak{h}_\beta(y_\beta) = \mathfrak{h}_\beta(1-y_\beta) = \max_{y\in[0,1]} \mathfrak{h}_\beta(y) > 1$ and $\lim_{\beta\to 0^+} y_\beta = 0$. Moreover $y_\beta = 1/2$ if and only if $\beta \geq q-1$.*
(iii) *If $q \in [2, 3)$, then:*
  (a) *for $\beta \in (0, 2^{q-1}-1)$, $\mathfrak{h}_\beta(y) \leq 1$ and the equality holds just in the endpoints $y_\beta = 0, 1$;*

(b) *for* $\beta = 2^{q-1} - 1$, $\mathfrak{h}_\beta(y) \le 1$ *and, in particular,*

$$\mathfrak{h}_\beta(y) = 1 \quad in \quad \begin{cases} 0, 1/2, 1 & if\ q \in (2,3) \\ [0,1] & if\ q = 2; \end{cases}$$

(c) $\beta > 2^{q-1} - 1$, *then* $\mathfrak{h}_\beta$ *achieves its unique global maximum on* $y_\beta = 1/2$ *and* $\mathfrak{h}_\beta(1/2) > 1$.

3. Existence and nonexistence results

In this section we prove the nonexistence result stated in Theorem 1.2 and the existence of a nontrivial radial ground state of ($\mathcal{S}_{\lambda,\beta}$), i.e. the first part of Theorem 1.1.

3.1. A Pohozaev identity and the nonexistence result

As it is usual for elliptic equations, the solutions satisfy a suitable identity called *Pohozaev identity*. It can be obtained, at least formally, by the relation

$$\frac{d}{dt} I_{\lambda,\beta}(u_t, v_t)\Big|_{t=1} = 0 \quad \text{where} \quad u_t(x) := u(x/t).$$

In the next lemma we get it rigorously. The proof is indeed standard, however we revise the argument for the sake of completeness. In what follows B_R stands for the ball centered in $0 \in \mathbb{R}^3$ and radius $R > 0$.

Lemma 3.1. *If* (u, v, ϕ) *is a solution of* (2.6) *with* $(u, v) \in \mathrm{H} \cap (L^{2q}(\mathbb{R}^3) \times L^{2q}(\mathbb{R}^3)) \cap (L^\infty_{\mathrm{loc}}(\mathbb{R}^3) \times L^\infty_{\mathrm{loc}}(\mathbb{R}^3))$, *with* fixed sign *if* $q \in [1/2, 1]$, *then it satisfies the Pohozaev identity*

$$\begin{aligned} &\frac{1}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{3}{2}(\|u\|_2^2 + \|v\|_2^2) + \frac{5}{4}\lambda \int (u^2 + v^2)\phi \\ &\quad = \frac{3}{2q}\left(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2\beta \int |u|^q |v|^q\right). \end{aligned} \tag{3.1}$$

Proof. Let (u, v, ϕ) be a solution of (2.6). If $q \in [1/2, 1]$, without loss of generality, we can assume $u, v > 0$.
Preliminarily we recall (see also [3, Proposition 2.1] and [8, Lemma 3.1]) that for any $R > 0$

$$\int_{B_R} -\Delta u\, x \cdot \nabla u = -\frac{1}{2}\int_{B_R} |\nabla u|^2 - \frac{1}{R}\int_{\partial B_R} |x \cdot \nabla u|^2 + \frac{R}{2}\int_{\partial B_R} |\nabla u|^2, \tag{3.2}$$

$$\int_{B_R} \phi u\, x \cdot \nabla u = -\frac{1}{2}\int_{B_R} u^2 x \cdot \nabla\phi - \frac{3}{2}\int_{B_R} \phi u^2 + \frac{R}{2}\int_{\partial B_R} \phi u^2, \tag{3.3}$$

$$\int_{B_R} g(u)\, x \cdot \nabla u = -3\int_{B_R} G(u) + R\int_{\partial B_R} G(u), \tag{3.4}$$

$$\int_{B_R} |u|^{q-2}u|v|^q x\cdot\nabla u + |u|^q|v|^{q-2}vx\cdot\nabla v = -\frac{3}{q}\int_{B_R}|u|^q|v|^q + \frac{R}{q}\int_{\partial B_R}|u|^q|v|^q, \tag{3.5}$$

where $g:\mathbb{R}\to\mathbb{R}$ is a continuous function with primitive $G(s)=\int_0^s g(\tau)d\tau$.

Observe that all the previous integrals make sense due to the regularity of u, v, ϕ. In particular, since $u\in L^\infty_{\mathrm{loc}}(\mathbb{R})$, then $\Delta u\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3)$ for all $p\geq 1$, the integral in the left hand side of (3.2) is well defined.

Then, multiplying the first equation in (2.6) by $x\cdot\nabla u$, integrating on B_R, and taking into account (3.2), (3.3), (3.4), (3.5), we get

$$\begin{aligned}
&\frac{1}{2}\int_{B_R}|\nabla u|^2+\frac{3}{2}\int_{B_R}u^2+\frac{\lambda}{2}\int_{B_R}u^2x\cdot\nabla\phi+\frac{3\lambda}{2}\int_{B_R}\phi u^2-\frac{3}{2q}\int_{B_R}|u|^{2q}\\
&\quad+\beta\int_{B_R}|v|^q|u|^{q-2}u\,x\cdot\nabla u=-\frac{1}{R}\int_{\partial B_R}|x\cdot\nabla u|^2+\frac{R}{2}\int_{\partial B_R}|\nabla u|^2+\frac{R}{2}\int_{\partial B_R}u^2\\
&\quad+\frac{\lambda R}{2}\int_{\partial B_R}\phi u^2-\frac{R}{2q}\int_{\partial B_R}|u|^{2q}.
\end{aligned}\tag{3.6}$$

In a similar way, from the second equation in (2.6) we infer

$$\begin{aligned}
&\frac{1}{2}\int_{B_R}|\nabla v|^2+\frac{3}{2}\int_{B_R}v^2+\frac{\lambda}{2}\int_{B_R}v^2x\cdot\nabla\phi+\frac{3\lambda}{2}\int_{B_R}\phi v^2-\frac{3}{2q}\int_{B_R}|v|^{2q}\\
&\quad+\beta\int_{B_R}|u|^q|v|^{q-2}v\,x\cdot\nabla v=-\frac{1}{R}\int_{\partial B_R}|x\cdot\nabla v|^2+\frac{R}{2}\int_{\partial B_R}|\nabla v|^2+\frac{R}{2}\int_{\partial B_R}v^2\\
&\quad+\frac{\lambda R}{2}\int_{\partial B_R}\phi v^2-\frac{R}{2q}\int_{\partial B_R}|v|^{2q}
\end{aligned}\tag{3.7}$$

and, from the third one, multiplying by $x\cdot\nabla\phi$, we deduce

$$\frac{1}{2}\int_{B_R}|\nabla\phi|^2+4\pi\int_{B_R}(u^2+v^2)\,x\cdot\nabla\phi=-\frac{1}{R}\int_{\partial B_R}|x\cdot\nabla\phi|^2+\frac{R}{2}\int_{\partial B_R}|\nabla\phi|^2. \tag{3.8}$$

Then, summing up (3.6) and (3.7), taking into account (3.8) and (3.5) we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_{B_R}(|\nabla u|^2+|\nabla v|^2)+\frac{3}{2}\int_{B_R}(u^2+v^2)-\frac{\lambda}{16\pi}\int_{B_R}|\nabla\phi|^2+\frac{3\lambda}{2}\int_{B_R}(u^2+v^2)\phi\\
&\quad-\frac{3}{2q}\int_{B_R}(|u|^{2q}+|v|^{2q}+2\beta|u|^q|v|^q)
\end{aligned}$$

$$= -\frac{1}{R}\int\limits_{\partial B_R}(|x\cdot\nabla u|^2+|x\cdot\nabla v|^2)+\frac{R}{2}\int\limits_{\partial B_R}(|\nabla u|^2+|\nabla v|^2)$$
$$+\frac{R}{2}\int\limits_{\partial B_R}(u^2+v^2)+\frac{\lambda R}{2}\int\limits_{\partial B_R}\phi(u^2+v^2)-\frac{R}{2q}\int\limits_{\partial B_R}(|u|^{2q}+|v|^{2q})+\frac{\lambda}{8\pi R}\int\limits_{\partial B_R}|x\cdot\nabla\phi|^2$$
$$-\frac{R\lambda}{16\pi}\int\limits_{\partial B_R}|\nabla\phi|^2-\frac{\beta R}{q}\int\limits_{\partial B_R}|u|^q|v|^q.$$

Arguing as in [3, pag. 321], there exists a suitable sequence $R_n \to +\infty$ on which the right hand side above tends to zero. Thus, passing to the limit we deduce that

$$\frac{1}{2}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)+\frac{3}{2}(\|u\|_2^2+\|v\|_2^2)-\frac{\lambda}{16\pi}\|\nabla\phi\|_2^2+\frac{3}{2}\lambda\int(u^2+v^2)\phi$$
$$=\frac{3}{2q}\Big(\|u\|_{2q}^{2q}+\|v\|_{2q}^{2q}+2\beta\int|u|^q|v|^q\Big).$$

Hence, using (2.7), we achieve the conclusion. □

With the Pohozaev identity (3.1), we can show easily our nonexistence result. Indeed we have

Proof of Theorem 1.2. Let $(u,v)\in \mathrm{H}\cap(L^{2q}(\mathbb{R}^3)\times L^{2q}(\mathbb{R}^3))$ be a nontrivial solution of $(\mathcal{S}_{\lambda,\beta})$ for $q\in[1/2,1]\cup[3,+\infty[$. Using the Nehari identities (2.9) and (2.10) and the Pohzaev identity (3.1) we have

$$0=\|\nabla u\|_2^2+\|\nabla v\|_2^2+\|u\|_2^2+\|v\|_2^2+\lambda\int(u^2+v^2)\phi_{u,v}-\|u\|_{2q}^{2q}-\|v\|_{2q}^{2q}-2\beta\int|u|^q|v|^q$$
$$=\Big(1-\frac{q}{3}\Big)(\|\nabla u\|_2^2+\|\nabla v\|_2^2)+(1-q)(\|u\|_2^2+\|v\|_2^2)+\Big(1-\frac{5}{6}q\Big)\lambda\int(u^2+v^2)\phi_{u,v},$$

which is strictly negative for $q\ge 3$ or strictly positive for $q\le 1$ and so we reach a contradiction. □

3.2. *Existence of a radial ground state*

Here we find a radial ground state solution for our system $(\mathcal{S}_{\lambda,\beta})$. As we have stated in the Introduction, to get compactness we restrict ourselves to radial functions. Thus, from now on, we will consider H_r as functional space, even if several facts do not require symmetry assumptions.

We start showing that, as in [28, Lemma 2.1], the following properties hold.

Lemma 3.2. *Let $q\in(1,3)$ and $\{(u_n,v_n)\}\subset \mathrm{H}_\mathrm{r}$ be such that $(u_n,v_n)\rightharpoonup(u,v)$ in H_r as $n\to+\infty$. We have, as $n\to+\infty$,*

$$\phi_{u_n,v_n}\to\phi_{u,v}\ \textit{in}\ D_\mathrm{r}^{1,2}(\mathbb{R}^3), \tag{3.9}$$
$$\int(u_n^2+v_n^2)\phi_{u_n,v_n}\to\int(u^2+v^2)\phi_{u,v}, \tag{3.10}$$

$$\int |u_n|^q |v_n|^q \to \int |u|^q |v|^q. \tag{3.11}$$

Proof. Let us define on $D_{\mathrm{r}}^{1,2}(\mathbb{R}^3)$ the linear and continuous operators

$$T_n(w) := \int \nabla w \nabla \phi_{u_n,v_n} \left(= 4\pi \int (u_n^2 + v_n^2) w \right),$$
$$T(w) := \int \nabla w \nabla \phi_{u,v} \left(= 4\pi \int (u^2 + v^2) w \right).$$

Then, due to the compact embedding of the radial functions we have

$$|T_n(w) - T(w)| \le 4\pi \|w\|_6 \Big(\|u_n^2 - u^2\|_{6/5} + \|v_n^2 - v^2\|_{6/5} \Big) \le \varepsilon_n \|\nabla w\|_2.$$

Hence $T_n - T \to 0$ as operators on $D_{\mathrm{r}}^{1,2}(\mathbb{R}^3)$, and by the Riesz Theorem this implies (3.9). Convergence (3.10) follows from

$$\phi_{u_n,v_n} \to \phi_{u,v} \text{ in } L^6(\mathbb{R}^3) \quad \text{and} \quad u_n^2 + v_n^2 \to u^2 + v^2 \text{ in } L^{6/5}(\mathbb{R}^3).$$

Finally, to get (3.11), we observe that, using again the compact embedding of the radial functions,

$$\| |u_n|^q - |u|^q \|_2, \quad \| |v_n|^q - |v|^q \|_2 \to 0.$$

Thus

$$\begin{aligned} \left| \int |u_n|^q |v_n|^q - |u|^q |v|^q \right| &\le \int |u_n|^q \Big| |v_n|^q - |v|^q \Big| + \int |v|^q \Big| |u_n|^q - |u|^q \Big| \\ &\le \|u_n\|_{2q}^q \| |v_n|^q - |v|^q \|_2 + \|v\|_{2q}^q \| |u_n|^q - |u|^q \|_2 = \varepsilon_n, \end{aligned}$$

concluding the proof. □

Let us consider now the *Nehari-Pohozaev manifold*

$$\mathcal{M} := \left\{ (u, v) \in \mathrm{H}_{\mathrm{r}} \setminus \{0\} : J_{\lambda,\beta}(u, v) = 0 \right\}$$

where

$$\begin{aligned} J_{\lambda,\beta}(u, v) := \frac{3}{2}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{1}{2}(\|u\|_2^2 + \|v\|_2^2) + \frac{3}{4}\lambda \int (u^2 + v^2)\phi_{u,v} \\ - \frac{4q-3}{2q} \left(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2\beta \int |u|^q |v|^q \right). \end{aligned}$$

Observe that the condition $J_{\lambda,\beta}(u, v) = 0$ can be obtained by a *linear combination* of the Nehari (2.9), (2.10) and Pohozaev (3.1) identities. Thus, $\mathcal{M}$ contains all nontrivial radial critical points of $I_{\lambda,\beta}$.

Moreover, the following simple result assures us that any couple $(u, v) \in \mathrm{H}_{\mathrm{r}} \setminus \{0\}$ can be uniquely projected on $\mathcal{M}$ via $\gamma_{u,v}$ (see its definition in (2.11)) and gives us a further property of such a projection.

Lemma 3.3. *For any* $(u, v) \in \mathrm{H_r} \setminus \{0\}$ *there exists a unique* $t_{u,v} > 0$ *such that* $\gamma_{u,v}(t_{u,v}) \in \mathcal{M}$ *and*

$$I_{\lambda,\beta}(\gamma_{u,v}(t_{u,v})) = \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)). \tag{3.12}$$

Proof. The existence and uniqueness of $t_{u,v}$ is an easy consequence of (a) in Lemma 2.3, since

$$J_{\lambda,\beta}(\gamma_{u,v}(t)) = \frac{3}{2}t^3(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \frac{t}{2}(\|u\|_2^2 + \|v\|_2^2) + \frac{3}{4}\lambda t^3 \int (u^2 + v^2)\phi_{u,v}$$
$$- \frac{4q-3}{2q}t^{4q-3}\left(\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2\beta \int |u|^q|v|^q\right)$$

and $q > 3/2$.
Moreover, since

$$J_{\lambda,\beta}(\gamma_{u,v}(t)) = t\frac{d}{dt}I_{\lambda,\beta}(\gamma_{u,v}(t)),$$

we have that $t_{u,v}$ is the unique strictly positive critical point of $I_{\lambda,\beta}(\gamma_{u,v}(t))$ and so, again by (a) in Lemma 2.3, we conclude. □

Now we are ready to find the ground state solutions of $(\mathcal{S}_{\lambda,\beta})$ by minimizing the functional $I_{\lambda,\beta}$ on $\mathcal{M}$.

Proof of Theorem 1.1 (existence of a ground state). We divide the proof in several steps.
Step 1: $\mathcal{M}$ *is bounded away from zero, i.e.* $(0, 0) \notin \partial\mathcal{M}$.
Let $(u, v) \in \mathcal{M}$. Since

$$2\int |u|^q|v|^q \le \|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} \le C\left(\|u\|^2 + \|v\|^2\right)^q$$

we deduce

$$\frac{1}{2}(\|u\|^2 + \|v\|^2) \le C\left(\|u\|^2 + \|v\|^2\right)^q,$$

so that there exists $\rho > 0$ such that $\|u\|^2 + \|v\|^2 \ge \rho > 0$ and the conclusion holds.
Step 2: $\mathfrak{m}_\beta := \inf_{\mathcal{M}} I_{\lambda,\beta} > 0$.
For $(u, v) \in \mathcal{M}$ we set, for simplicity,

$$\begin{cases} a := \|\nabla u\|_2^2 + \|\nabla v\|_2^2, & b := \|u\|_2^2 + \|v\|_2^2, \\ c := \lambda \int (u^2 + v^2)\phi_{u,v}, & d := \|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2\beta \int |u|^q|v|^q. \end{cases}$$

If $k := I_{\lambda,\beta}(u, v)$, we have

$$\begin{cases} \frac{1}{2}a+\frac{1}{2}b+\frac{1}{4}c-\frac{1}{2q}d=k \\ \frac{3}{2}a+\frac{1}{2}b+\frac{3}{4}c-\frac{4q-3}{2q}d=0. \end{cases}$$

In terms of a, b, k the unknown c is given by

$$0<c=2\frac{(4q-3)k-(2q-3)a-2b(q-1)}{2q-3}.$$

Then taking into account Step 1, we have

$$(2q-3)\rho<(2q-3)(a+b)<(2q-3)a+2(q-1)b<(4q-3)k, \tag{3.13}$$

where $\rho>0$ is the constant found at the end of the previous step, meaning that k is bounded away from zero.

Step 3: *If* $\{(u_n, v_n)\}$ *is a minimizing sequence for* $I_{\lambda,\beta}$ *on* $\mathcal{M}$*, then it is bounded. Hence, up to subsequence, it weakly converges to some* $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ *in* $\mathrm{H_r}$.

Let $\{(u_n, v_n)\}\subset\mathcal{M}$ such that $k_n := I_{\lambda,\beta}(u_n, v_n)\to\mathfrak{m}_\beta$. Setting for simplicity

$$\begin{cases} a_n := \|\nabla u_n\|_2^2+\|\nabla v_n\|_2^2, & b_n := \|u_n\|_2^2+\|v_n\|_2^2, \\ c_n := \lambda\int(u_n^2+v_n^2)\phi_{u_n,v_n}, & d_n := \|u_n\|_{2q}^{2q}+\|v_n\|_{2q}^{2q}+2\beta\int|u_n|^q|v_n|^q, \end{cases} \tag{3.14}$$

arguing as in Step 2, see (3.13), we get

$$(2q-3)(a_n+b_n)<(4q-3)k_n\to(4q-3)\mathfrak{m}_\beta$$

and so the minimizing sequence $\{(u_n, v_n)\}$ is bounded.

Step 4: $\{(u_n, v_n)\}$ *strongly converges to* $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ *in* $\mathrm{H_r}$. *Then* $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)\in\mathcal{M}$ *and it minimizes* $I_{\lambda,\beta}$.

Here is the scenario in which we need the radial setting.

Observe that, by the previous step, it follows that

$$u_n\rightharpoonup\mathfrak{u}_\beta,\ \ v_n\rightharpoonup\mathfrak{v}_\beta,\ \text{in } L^2(\mathbb{R}^3) \text{ and in } D^{1,2}(\mathbb{R}^3) \tag{3.15}$$

and, eventually passing to a suitable subsequence,

$$\|\nabla\mathfrak{u}_\beta\|_2^2\le\lim_n\|\nabla u_n\|_2^2,\quad \|\nabla\mathfrak{v}_\beta\|_2^2\le\lim_n\|\nabla v_n\|_2^2,\quad \|\mathfrak{u}_\beta\|_2^2\le\lim_n\|u_n\|_2^2,\quad \|\mathfrak{v}_\beta\|_2^2\le\lim_n\|v_n\|_2^2. \tag{3.16}$$

Maintaining the notations in (3.14), we define

$$\overline{a}:=\lim_n a_n\,,\ \ \overline{b}:=\lim_n b_n\,,\ \ \overline{c}:=\lim_n c_n\,,\ \ \overline{d}:=\lim_n d_n,$$

where we are assuming that the limits exist (eventually passing to suitable subsequences) being $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ bounded sequences (see the previous Step).

Observe also that, by Step 1,

$$\overline{a}+\overline{b}>0. \tag{3.17}$$

Moreover, the relations

$$I_{\lambda,\beta}(u_n,v_n)\to \mathfrak{m}_\beta \quad \text{and} \quad J_{\lambda,\beta}(u_n,v_n)=0$$

give

$$\begin{cases} \frac{1}{2}\overline{a}+\frac{1}{2}\overline{b}+\frac{1}{4}\overline{c}-\frac{1}{2q}\overline{d}=\mathfrak{m}_\beta \\ \frac{3}{2}\overline{a}+\frac{1}{2}\overline{b}+\frac{3}{4}\overline{c}-\frac{4q-3}{2q}\overline{d}=0. \end{cases} \tag{3.18}$$

Thus, by the second equation in (3.18) and (3.17) we get $\overline{d}>0$.
Hence, using an analogous notation as before for the pair $(\mathfrak{u}_\beta,\mathfrak{v}_\beta)$, namely

$$\begin{cases} a:=\|\nabla \mathfrak{u}_\beta\|_2^2+\|\nabla \mathfrak{v}_\beta\|_2^2, & b:=\|\mathfrak{u}_\beta\|_2^2+\|\mathfrak{v}_\beta\|_2^2, \\ c:=\lambda\int(\mathfrak{u}_\beta^2+\mathfrak{v}_\beta^2)\phi_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}, & d:=\|\mathfrak{u}_\beta\|_{2q}^{2q}+\|\mathfrak{v}_\beta\|_{2q}^{2q}+2\beta\int|\mathfrak{u}_\beta|^q|\mathfrak{v}_\beta|^q, \end{cases} \tag{3.19}$$

by (3.16), we have

$$a\le\overline{a} \quad \text{and} \quad b\le\overline{b}. \tag{3.20}$$

Observe that, due to Lemma 3.2 and to the compact embedding in the radial setting

$$\overline{c}=c \quad \text{and} \quad \overline{d}=d.$$

If $a+b<\overline{a}+\overline{b}$, then, taking into account that $J_{\lambda,\beta}(u_n,v_n)=0$, we have that $J_{\lambda,\beta}(\mathfrak{u}_\beta,\mathfrak{v}_\beta)<0$, meaning that $(\mathfrak{u}_\beta,\mathfrak{v}_\beta)\notin\mathcal{M}$ and that $(\mathfrak{u}_\beta,\mathfrak{v}_\beta)\neq(0,0)$. This implies that $a,b,c,d>0$ and, by (3.20), also $\overline{a},\overline{b}>0$.
Moreover, by Lemma 3.3 there exists a unique $t_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}>0$ such that $\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t_{\mathfrak{u}_\beta,\mathfrak{v}_\beta})\in\mathcal{M}$ (see (2.11)).
Consider now, for $t\ge 0$, the functions

$$f(t)=\frac{t^3}{2}a+\frac{t}{2}b+\frac{t^3}{4}c-\frac{t^{4q-3}}{2q}d, \qquad \overline{f}(t)=\frac{t^3}{2}\overline{a}+\frac{t}{2}\overline{b}+\frac{t^3}{4}\overline{c}-\frac{t^{4q-3}}{2q}\overline{d}.$$

Note that

$$f(t)=I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t)) \quad \text{and} \quad tf'(t)=J_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t)).$$

The functions f and $\overline{f}$ have both a unique critical point corresponding to the global maximum (see (a) in Lemma 2.3). In particular, the global maximizer of f is $t_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}$ and, by (3.18), we deduce that $\overline{f}$ achieves the maximum in $t=1$. Moreover, since we are assuming $a+b<\overline{a}+\overline{b}$, it holds $f(t)<\overline{f}(t)$ for $t>0$. Hence $\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t_{\mathfrak{u}_\beta,\mathfrak{v}_\beta})\in\mathcal{M}$ and

$$I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t_{\mathfrak{u}_\beta,\mathfrak{v}_\beta})) = f(t_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}) < \max_{t\geq 0} \overline{f}(t) = \mathfrak{m}_\beta,$$

which is a contradiction.
Hence, by (3.20), we infer $a = \overline{a}$ and $b = \overline{b}$, so that, using (3.15), we get $(u_n, v_n) \to (\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ in $\mathrm{H_r}$.
Step 5: $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ *is a regular point of* $\mathcal{M}$*, i.e.* $J'_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{v}_\beta) \neq 0$.
Assume by contradiction that $J'_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{v}_\beta) = 0$ so that we have

$$\begin{cases} -3\Delta\mathfrak{u}_\beta + \mathfrak{u}_\beta + 3\lambda\phi_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}\mathfrak{u}_\beta - (4q-3)\left(|\mathfrak{u}_\beta|^{2q-2} + \beta|\mathfrak{u}_\beta|^{q-2}|\mathfrak{v}_\beta|^q\right)\mathfrak{u}_\beta = 0 \\ -3\Delta\mathfrak{v}_\beta + \mathfrak{v}_\beta + 3\lambda\phi_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}\mathfrak{v}_\beta - (4q-3)\left(|\mathfrak{v}_\beta|^{2q-2} + \beta|\mathfrak{v}_\beta|^{q-2}|\mathfrak{u}_\beta|^q\right)\mathfrak{v}_\beta = 0. \end{cases} \tag{3.21}$$

Then, under the notations (3.19), we have

$$\begin{cases} \frac{1}{2}a + \frac{1}{2}b + \frac{1}{4}c - \frac{1}{2q}d = \mathfrak{m}_\beta, \\ \frac{3}{2}a + \frac{1}{2}b + \frac{3}{4}c - \frac{4q-3}{2q}d = 0, \\ 3a + b + 3c - (4q-3)d = 0, \\ \frac{3}{2}a + \frac{3}{2}b + \frac{15}{4}c - 3\frac{4q-3}{2q}d = 0, \end{cases}$$

where the third equation is simply $J'_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{v}_\beta)[\mathfrak{u}_\beta, \mathfrak{v}_\beta] = 0$ and, finally, the fourth equation is the Pohozaev identity for (3.21). The solution of the above system is given by

$$a = -\frac{4q-3}{4(2q-3)}\mathfrak{m}_\beta, \quad b = 3\frac{4q-3}{4(q-1)}\mathfrak{m}_\beta, \quad c = -\frac{4q-3}{2(2q-3)}\mathfrak{m}_\beta, \quad d = -\frac{3q}{4(2q-3)(q-1)}\mathfrak{m}_\beta.$$

Since $q \in (3/2, 3)$, then $a < 0$, which is impossible.
Step 6: $I'_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{v}_\beta) = 0$.
Thanks to the Lagrange multiplier rule we know that, for some $\ell \in \mathbb{R}$,

$$I'_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{v}_\beta) = \ell J'_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{v}_\beta).$$

We want to show that $\ell = 0$.
By expliciting the above identity we get

$$\begin{aligned} &-(3\ell-1)\Delta\mathfrak{u}_\beta + (\ell-1)\mathfrak{u}_\beta + (3\ell-1)\lambda\phi_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}\mathfrak{u}_\beta \\ &\quad -((4q-3)\ell-1)\left[|\mathfrak{u}_\beta|^{2q-2} + \beta|\mathfrak{u}_\beta|^{q-2}|\mathfrak{v}_\beta|^q\right]\mathfrak{u}_\beta = 0 \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &-(3\ell-1)\Delta\mathfrak{v}_\beta + (\ell-1)\mathfrak{v}_\beta + (3\ell-1)\lambda\phi_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}\mathfrak{v}_\beta \\ &\quad -((4q-3)\ell-1)\left[|\mathfrak{v}_\beta|^{2q-2} + \beta|\mathfrak{v}_\beta|^{q-2}|\mathfrak{u}_\beta|^q\right]\mathfrak{v}_\beta = 0. \end{aligned} \tag{3.23}$$

Now, multiplying (3.22) and (3.23) by $\mathfrak{u}_\beta$ and $\mathfrak{v}_\beta$ respectively, integrating, and, finally, summing up the two identities obtained, we get, using again the notations in (3.19),

$$(3\ell-1)a+(\ell-1)b+(3\ell-1)c-((4q-3)\ell-1)d=0.$$

On the other hand, arguing as in Lemma 3.1, we can associate to (3.22) and (3.23) the Pohozaev identity

$$\frac{3\ell-1}{2}a+\frac{3}{2}(\ell-1)b+\frac{5}{4}(3\ell-1)c-\frac{3}{2q}((4q-3)\ell-1)d=0.$$

Then a, b, c, d satisfy the system

$$\begin{cases} \frac{1}{2}a+\frac{1}{2}b+\frac{1}{4}c-\frac{1}{2q}d=\mathfrak{m}_\beta, \\ \frac{3}{2}a+\frac{1}{2}b+\frac{3}{4}c-\frac{4q-3}{2q}d=0, \\ (3\ell-1)a+(\ell-1)b+(3\ell-1)c-((4q-3)\ell-1)d=0, \\ \frac{3\ell-1}{2}a+\frac{3}{2}(\ell-1)b+\frac{5}{4}(3\ell-1)c-\frac{3}{2q}((4q-3)\ell-1)d=0. \end{cases}$$

The determinant of the matrix of the coefficients is

$$-\frac{\ell(3\ell-1)(q-1)(2q-3)}{q}.$$

The assumptions on q imply that $q-1\neq 0$ and $2q-3\neq 0$. Moreover, also $\ell\neq 1/3$. Indeed, if it were $\ell=1/3$, the third equation of the system above would be

$$-\frac{2}{3}b-\frac{2(2q-3)}{3}d=0$$

which is impossible since $b, d>0$. Thus, if it were also $\ell\neq 0$, then the determinant would be different from zero, meaning that the system would have a unique solution. In particular

$$d=-\frac{3q}{4(q-1)(2q-3)}\mathfrak{m}_\beta<0$$

which is impossible. Summing up it yields $\ell=0$, concluding the proof of the Step. □

Remark 3.4. Without loss of generality, since $(|\mathfrak{u}_\beta|, |\mathfrak{v}_\beta|)$ is also a solution at the level $\mathfrak{m}_\beta$, applying the Maximum Principle, we can assume that, whenever $\mathfrak{u}_\beta, \mathfrak{v}_\beta$ are nontrivial, they are strictly positive.

Remark 3.5. For future reference, we observe that the statements of previous Steps 5 and 6 and the inequality

$$0 < \|\mathfrak{u}_\beta\|^2 + \|\mathfrak{v}_\beta\|^2 \le \frac{4q-3}{2q-3}\mathfrak{m}_\beta,$$

which follows by (3.13) in Step 2, hold for any pair of ground states.

Moreover, as an immediate consequence of Theorem 1.1 what we have just seen we can prove the following further result.

Corollary 3.6. *Let* $(\mathfrak{u}_\beta, \mathfrak{v}_\beta) \in \mathrm{H_r} \setminus \{0\}$ *be a ground state found in Theorem 1.1. We have that*

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(1)) = \inf_{(u,v)\in \mathrm{H_r}\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t))$$

Proof. By Lemma 3.3 we have that, for every $(u, v) \in \mathrm{H_r} \setminus \{0\}$,

$$\mathfrak{m}_\beta = \min_{(u,v)\in\mathcal{M}} I_{\lambda,\beta}(u,v) \le I_{\lambda,\beta}(\gamma_{u,v}(t_{u,v})) = \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)).$$

Then, passing to the infimum on $(u, v) \in \mathrm{H_r} \setminus \{0\}$, we get

$$\mathfrak{m}_\beta \le \inf_{(u,v)\in \mathrm{H_r}\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)) \le \max_{t>0} I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t)) = I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(1)) = \mathfrak{m}_\beta$$

concluding the proof. □

We conclude this section showing a further interesting property.

By Remark 3.4, in polar form the ground state $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ can be written as

$$(\mathfrak{u}_\beta, \mathfrak{v}_\beta) = (\varrho_\beta \cos\vartheta_\beta, \varrho_\beta \sin\vartheta_\beta), \quad \varrho_\beta^2 = \mathfrak{u}_\beta^2 + \mathfrak{v}_\beta^2 > 0, \quad \vartheta_\beta = \vartheta_\beta(x) \in [0, \pi/2]. \tag{3.24}$$

Note that whenever the ground state is vectorial, then $\vartheta_\beta \in (0, \pi/2)$, while for semitrivial ground states it is $\vartheta_\beta \equiv 0$ or $\vartheta_\beta \equiv \pi/2$.

The next lemma shows how, starting from $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ we can obtain a convenient ground state with the additional property of having as angular coordinate a constant function θ_β.

By Lemma 2.4, let $y_\beta = \cos^2\theta_\beta \in [0, 1/2]$ be a maximum point of $\mathfrak{h}_\beta$.

Lemma 3.7. *For* $\beta \ge 0$*, there exists* $t_\beta > 0$ *such that* $\gamma_{\varrho_\beta\cos\theta_\beta,\varrho_\beta\sin\theta_\beta}(t_\beta) \in \mathcal{M}$ *and*

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\gamma_{\varrho_\beta\cos\theta_\beta,\varrho_\beta\sin\theta_\beta}(t_\beta)). \tag{3.25}$$

In particular $\gamma_{\varrho_\beta\cos\theta_\beta,\varrho_\beta\sin\theta_\beta}(t_\beta)$ *is a ground state solution.*

Proof. The conclusion will be achieved showing that the *projection* of $(\varrho_\beta\cos\theta_\beta, \varrho_\beta\sin\theta_\beta)$ in $\mathcal{M}$ reaches the ground state level.

Since $(\varrho_\beta\cos\theta_\beta, \varrho_\beta\sin\theta_\beta) \in \mathrm{H_r} \setminus \{0\}$, by Lemma 3.3 there exists a unique $t_\beta > 0$ such that

$$\gamma_{\varrho_\beta\cos\theta_\beta,\varrho_\beta\sin\theta_\beta}(t_\beta) \in \mathcal{M}.$$

Let us show it is at the ground state level. Observe that

$$\mathfrak{m}_\beta = \inf_{(u,v)\in \mathrm{H_r}\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)) \leq \max_{t>0} I_{\lambda,\beta}(\gamma_{\varrho_\beta \cos\theta_\beta, \varrho_\beta \sin\theta_\beta}(t)) = I_{\lambda,\beta}(\gamma_{\varrho_\beta \cos\theta_\beta, \varrho_\beta \sin\theta_\beta}(t_\beta)).$$

Moreover, being $y_\beta = \cos^2\theta_\beta$ a maximum point of $\mathfrak{h}_\beta$ in $[0, 1/2]$,

$$\mathfrak{u}_\beta^{2q} + \mathfrak{v}_\beta^{2q} + 2\beta \mathfrak{u}_\beta^q \mathfrak{v}_\beta^q = \mathfrak{h}_\beta(\cos^2\vartheta_\beta)\varrho_\beta^{2q} \leq \mathfrak{h}_\beta(y_\beta)\varrho_\beta^{2q}.$$

Then, since

$$\|\nabla \mathfrak{u}_\beta\|_2^2 + \|\nabla \mathfrak{v}_\beta\|_2^2 = \|\varrho_\beta \nabla\vartheta_\beta\|_2^2 + \|\nabla\varrho_\beta\|_2^2 \geq \|\nabla\varrho_\beta\|_2^2,$$

for every $t > 0$,

$$\begin{aligned} I_{\lambda,\beta}(\gamma_{\varrho_\beta \cos\theta_\beta, \varrho_\beta \sin\theta_\beta}(t)) &= \frac{t^3}{2}\|\nabla\varrho_\beta\|_2^2 + \frac{t}{2}\|\varrho_\beta\|_2^2 + \frac{\lambda}{4}t^3 \int \phi_{\varrho_\beta}\varrho_\beta^2 - \frac{t^{4q-3}}{2q}\mathfrak{h}_\beta(y_\beta)\|\varrho_\beta\|_{2q}^{2q} \\ &\leq \frac{t^3}{2}\|\nabla\varrho_\beta\|_2^2 + \frac{t}{2}\|\varrho_\beta\|_2^2 + \frac{\lambda}{4}t^3 \int \phi_{\varrho_\beta}\varrho_\beta^2 \\ &\quad - \frac{t^{4q-3}}{2q}\int \left(\mathfrak{u}_\beta^{2q} + \mathfrak{v}_\beta^{2q} + 2\beta \mathfrak{u}_\beta^q \mathfrak{v}_\beta^q\right) \\ &\leq I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t)). \end{aligned}$$

Thus, by Lemma 3.3,

$$I_{\lambda,\beta}(\gamma_{\varrho_\beta \cos\theta_\beta, \varrho_\beta \sin\theta_\beta}(t_\beta)) \leq I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t_\beta)) \leq \mathfrak{m}_\beta,$$

concluding the proof. □

Remark 3.8. Whenever $\max_{[0,1]}\mathfrak{h}_\beta > 1$, then $y_\beta \in (0, 1/2]$ and Lemma 3.7 gives a vectorial ground state.

4. The case $\beta = 0$

In this section we prove item (i) of Theorem 1.1: if $\beta = 0$, each ground state solution $(\mathfrak{u}_0, \mathfrak{v}_0)$ of $(\mathcal{S}_{\lambda,\beta})$ is semitrivial.

Consider system $(\mathcal{S}_{\lambda,\beta})$ for $\beta = 0$, namely

$$\begin{cases} -\Delta u + u + \lambda\phi_{u,v}u = |u|^{2q-2}u \\ -\Delta v + v + \lambda\phi_{u,v}v = |v|^{2q-2}v \end{cases} \quad \text{in } \mathbb{R}^3. \tag{4.1}$$

Of course $(\mathfrak{w}, 0)$ and $(0, \mathfrak{w})$, where $\mathfrak{w}$ is a ground state of (1.4) obtained in [28] (see also Section 2), are solutions of (4.1). Since for every $u \in H_r^1(\mathbb{R}^3)$, we have that

$$\mathcal{I}_{\lambda,0}(u) = I_{\lambda,0}(u, 0) = I_{\lambda,0}(0, u),$$

then, by (2.4),

$$\mathfrak{n} = \mathcal{I}_{\lambda,0}(\mathfrak{w}) = I_{\lambda,0}(\mathfrak{w},0) = I_{\lambda,0}(0,\mathfrak{w}) = \inf_{u\in H^1_{\rm r}(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} I_{\lambda,0}(\zeta_u(t),0),$$

where the path ζ_u has been defined in (2.2).
Moreover

$$\mathfrak{m}_0 \le \mathfrak{n}. \tag{4.2}$$

First we prove that the ground state level of the two variables functional $I_{\lambda,0}$ is the same of the ground state level of the one variable functional $\mathcal{I}_{\lambda,0}$.

Lemma 4.1. $\mathfrak{m}_0 = \mathfrak{n}$.

Proof. If we use the *polar coordinates* for the couples (u,v), namely we write

$$(u,v) = (\varrho\cos\vartheta, \varrho\sin\vartheta) \text{ where } \varrho^2 = u^2+v^2 \text{ and } \vartheta = \vartheta(x) \in [0,2\pi],$$

we have that

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 = \|\varrho\nabla\vartheta\|_2^2 + \|\nabla\varrho\|_2^2$$

and, by (i) in Lemma 2.4,

$$\|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} = \int \varrho^{2q}\left(\cos^{2q}\vartheta + \sin^{2q}\vartheta\right) \le \|\varrho\|_{2q}^{2q}.$$

Then, for every $t>0$,

$$I_{\lambda,0}(\gamma_{u,v}(t)) \ge \frac{t^3}{2}\|\varrho\nabla\vartheta\|_2^2 + \frac{t^3}{2}\|\nabla\varrho\|_2^2 + \frac{t}{2}\|\varrho\|_2^2 + \frac{\lambda}{4}t^3\int\phi_\varrho\varrho^2 - \frac{t^{4q-3}}{2q}\|\varrho\|_{2q}^{2q} \ge \mathcal{I}_{\lambda,0}(\zeta_\varrho(t)). \tag{4.3}$$

Hence, (4.3), (2.4), and (4.2) imply

$$\mathfrak{m}_0 = \inf_{(u,v)\in \mathrm{H}_{\rm r}\setminus\{0\}} \max_{t>0} I_{\lambda,0}(\gamma_{u,v}(t)) \ge \inf_{\varrho\in H^1_{\rm r}(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} I_{\lambda,0}(\zeta_\varrho(t),0) = \mathfrak{n},$$

concluding the proof. □

Now we are ready to show the main goal of this section.

Proof of (i) of Theorem 1.1. Assume by contradiction that there exists a vectorial ground state $(\overline{u},\overline{v})$. By Remark 3.4, without loss of generality we can assume that $\overline{u},\overline{v}>0$. Thus, using as before the *polar coordinates*, we can write

$$(\overline{u},\overline{v}) = (\overline{\varrho}\cos\overline{\vartheta}, \overline{\varrho}\sin\overline{\vartheta}), \text{ with } \overline{\varrho}^2 = \overline{u}^2 + \overline{v}^2 \text{ and } \overline{\vartheta} = \overline{\vartheta}(x) \in (0,\pi/2).$$

Then, using (i) in Lemma 2.4, we have that $\cos^{2q}\overline{\vartheta} + \sin^{2q}\overline{\vartheta} < 1$, and so by (3.12) and arguing as in (4.3), we get that, for all $t>0$,

$$\mathfrak{m}_0 \geq I_{\lambda,0}(\gamma_{\overline{u},\overline{v}}(t)) > I_{\lambda,0}(\zeta_{\overline{\varrho}}(t), 0).$$

Then

$$\mathfrak{m}_0 > \max_{t>0} I_{\lambda,0}(\zeta_{\overline{\varrho}}(t), 0) \geq \inf_{\varrho\in H_r^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} I_{\lambda,0}(\zeta_{\varrho}(t), 0) = \mathfrak{n},$$

which is a contradiction with Lemma 4.1. □

5. The case $\beta > 0$ and small

In this section we consider

$$\beta \in \begin{cases} (0, 2^{q-1} - 1) & \text{for } q \in [2, 3) \quad \text{as in (ii) of Theorem 1.1,} \\ (0, q - 1) & \text{for } q \in (3/2, 2) \quad \text{as in (iii) of Theorem 1.1.} \end{cases}$$

Let us start with the proof of item (ii) of Theorem 1.1.
Preliminarily, as in Section 4, we prove

Lemma 5.1. *If* $\beta \in (0, 2^{q-1} - 1]$ *and* $q \in [2, 3)$*, then* $\mathfrak{m}_\beta = \mathfrak{n}$.

Proof. Since for any $u \in H_r^1(\mathbb{R}^3)$ it holds

$$\mathcal{I}_{\lambda,0}(u) = I_{\lambda,\beta}(u, 0) = I_{\lambda,\beta}(0, u), \tag{5.1}$$

then

$$\mathfrak{n} = \mathcal{I}_{\lambda,0}(\mathfrak{w}) = \inf_{u\in H_r^1(\mathbb{R}^3)\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\zeta_u(t), 0) \geq \inf_{(u,v)\in \mathrm{H}_r\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)) = \mathfrak{m}_\beta.$$

Moreover, introducing the polar coordinates as in Lemma 4.1 and using (a) and (b) of (iii) in Lemma 2.4 we get

$$\begin{aligned} \|u\|_{2q}^{2q} + \|v\|_{2q}^{2q} + 2\beta \int |u|^q |v|^q &= \int \varrho^{2q} \left(\cos^{2q}\vartheta + \sin^{2q}\vartheta + 2\beta |\cos\vartheta|^q (1 - \sin^2\vartheta)^{q/2} \right) \\ &\leq \|\varrho\|_{2q}^{2q}. \end{aligned}$$

Thus, arguing as in Lemma 4.1, we arrive at $I_{\lambda,\beta}(\gamma_{u,v}(t)) \geq \mathcal{I}_{\lambda,0}(\zeta_\rho(t))$ and so $\mathfrak{m}_\beta \geq \mathfrak{n}$. □

As an immediate consequence we can conclude as follows.

Proof of (ii) in Theorem 1.1. Hence for $\beta \in (0, 2^{q-1} - 1)$, the proof is completely analogous to that one of item (i) of Theorem 1.1, using (a) of (iii) in Lemma 2.4 instead if (i). □

Let us address now (iii) of Theorem 1.1. We first show that the ground state $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ is vectorial and then that it converges to a semitrivial solution as $\beta \to 0^+$.

First we show a preliminary property that we will use also in the next section, since the conclusion holds whenever there exists a point where $\mathfrak{h}_\beta$ introduced in Lemma 2.4 is greater than 1.

Lemma 5.2. *If* $\beta \in (0, q-1)$ *and* $q \in (3/2, 2)$*, then*

$$\mathfrak{m}_\beta < \mathfrak{n}. \tag{5.2}$$

Proof. In virtue of (ii) of Lemma 2.4, let $\theta_\beta \in (0, \pi/2)$ be defined by $y_\beta = \cos^2\theta_\beta \in (0, 1/2)$ and consider the vectorial function

$$(\widetilde{u}, \widetilde{v}) := (\mathfrak{w}\cos\theta_\beta, \mathfrak{w}\sin\theta_\beta) \in \mathrm{H_r} \setminus \{0\}.$$

A simple computation shows that

$$\begin{aligned}
\|\nabla\widetilde{u}\|_2^2 + \|\nabla\widetilde{v}\|_2^2 &= \|\nabla\mathfrak{w}\|_2^2 \\
\|\widetilde{u}\|_2^2 + \|\widetilde{v}\|_2^2 &= \|\mathfrak{w}\|_2^2, \\
\int (\widetilde{u}^2 + \widetilde{v}^2)\phi_{\widetilde{u},\widetilde{v}} &= \int \phi_{\mathfrak{w}}\mathfrak{w}^2, \\
\|\widetilde{u}\|_{2q}^{2q} + \|\widetilde{v}\|_{2q}^{2q} + 2\beta\int |\widetilde{u}|^q|\widetilde{v}|^q &= \left(\cos^{2q}\theta_\beta + \sin^{2q}\theta_\beta + 2\beta\cos^q\theta_\beta\sin^q\theta_\beta\right)\|\mathfrak{w}\|_{2q}^{2q} \\
&= \left(y_\beta^q + (1-y_\beta)^q + 2\beta y_\beta^{q/2}(1-y_\beta)^{q/2}\right)\|\mathfrak{w}\|_{2q}^{2q} \\
&> \|\mathfrak{w}\|_{2q}^{2q},
\end{aligned}$$

where the last inequality is due again to (ii) of Lemma 2.4.
Consequently, recalling (5.1), for any $t > 0$ we have

$$\begin{aligned}
\mathcal{I}_{\lambda,0}(\zeta_{\mathfrak{w}}(t)) = I_{\lambda,\beta}(\gamma_{\mathfrak{w},0}(t)) &= \frac{t^3}{2}\|\nabla\mathfrak{w}\|_2^2 + \frac{t}{2}\|\mathfrak{w}\|_2^2 + \frac{\lambda}{4}t^3\int\phi_{\mathfrak{w}}\mathfrak{w}^2 - \frac{t^{4q-3}}{2q}\|\mathfrak{w}\|_{2q}^{2q} \\
&> \frac{t^3}{2}(\|\nabla\widetilde{u}\|_2^2 + \|\nabla\widetilde{v}\|_2^2) + \frac{t}{2}(\|\widetilde{u}\|_2^2 + \|\widetilde{v}\|_2^2) + \frac{\lambda}{4}t^3\int\phi_{\widetilde{u},\widetilde{v}}(\widetilde{u}^2 + \widetilde{v}^2) \\
&\quad - \frac{t^{4q-3}}{2q}\left(\|\widetilde{u}\|_{2q}^{2q} + \|\widetilde{v}\|_{2q}^{2q} + 2\beta\int|\widetilde{u}|^q|\widetilde{v}|^q\right) \\
&= I_{\lambda,\beta}(\gamma_{\widetilde{u},\widetilde{v}}(t)).
\end{aligned}$$

Passing to the maximum on $t > 0$, since both maxima are achieved, and recalling that $t \mapsto \mathcal{I}_{\lambda,0}(\zeta_{\mathfrak{w}}(t))$ achieves its maximum in $t = 1$ being $\mathfrak{w} \in \mathcal{N}^\lambda$ (see (2.1)), we can write

$$\mathfrak{n} = \mathcal{I}_{\lambda,0}(\mathfrak{w}) > \max_{t>0} I_{\lambda,\beta}(\gamma_{\widetilde{u},\widetilde{v}}(t)) \geq \inf_{(u,v)\in\mathrm{H_r}\setminus\{0\}} \max_{t>0} I_{\lambda,\beta}(\gamma_{u,v}(t)) = \mathfrak{m}_\beta$$

concluding the proof. □

We point out that, in contrast to the proof of Lemma 3.7, where we need to take exactly y_β, here in Lemma 5.2 it is enough to take an arbitrary point where $\mathfrak{h}_\beta$ is greater than one.

As an immediate consequence of Lemma 5.2 we have

Proof of (iii) in Theorem 1.1 (vectorial ground state). If the ground state were for instance of type $(\mathfrak{u}_\beta, 0)$, then, recalling (5.1), we would have

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\mathfrak{u}_\beta, 0) = \mathcal{I}_{\lambda,0}(\mathfrak{u}_\beta) \geq \mathfrak{n}$$

contradicting (5.2). □

As observed in Remark 3.8, by Lemma 3.7, if $y_\beta = \cos^2\theta_\beta \in (0, 1/2]$ is the maximum point of $\mathfrak{h}_\beta$, we have

Corollary 5.3. *If $\beta \in (0, q-1)$ and $q \in (3/2, 2)$, then there exists $t_\beta > 0$ such that $\gamma_{\varrho_\beta \cos\theta_\beta, \varrho_\beta \sin\theta_\beta}(t_\beta)$ is a vectorial ground state.*

Now we show the *asymptotic behavior* as $\beta \to 0^+$ of the vectorial ground state solutions found in (iii) of Theorem 1.1. As in the proof of (i) of Theorem 1.1, we assume without loss of generality that $\mathfrak{u}_\beta$ and $\mathfrak{v}_\beta$ are positive.

Arguing as in Step 1 and Step 2 of the Proof of Theorem 1.1, we can get for $\beta > 0$ in a bounded set a uniform lower bound for the ground states levels $\mathfrak{m}_\beta$.
However, in what follows, we give an estimate of such a lower bound depending on the energy level of the ground state $\mathfrak{g}$ of

$$-\Delta u + u = |u|^{2q-2}u \text{ in } \mathbb{R}^3$$

(see e.g. [36]).
To this aim let us introduce another limit problem which will be useful for our purpose: system $(\mathcal{S}_{\lambda,\beta})$ with $\lambda = 0$, namely

$$\begin{cases} -\Delta u + u = |u|^{2q-2}u + \beta|v|^q|u|^{q-2}u \\ -\Delta v + v = |v|^{2q-2}v + \beta|u|^q|v|^{q-2}v \end{cases} \text{in } \mathbb{R}^3. \tag{5.3}$$

Let $(\widehat{\mathfrak{u}}_\beta, \widehat{\mathfrak{v}}_\beta) \in \mathrm{H_r}$ be the vectorial, positive and radial ground state solution, see [22, Corollary 1], which exists for any $\beta > 0$ and $q \in (3/2, 2)$. In our notations, the energy functional related to (5.3) is $I_{0,\beta}$. Since

$$I_{0,\beta}(\gamma_{\widehat{\mathfrak{u}}_\beta, \widehat{\mathfrak{v}}_\beta}(t)) \to -\infty \quad \text{as } t \to +\infty,$$

there exists $a_\beta > 0$ such that

$$\gamma_{\widehat{\mathfrak{u}}_\beta, \widehat{\mathfrak{v}}_\beta} \in \Gamma_\beta := \left\{\eta \in C([0, a_\beta], \mathrm{H_r}) : I_{0,\beta}(\eta(0)) = 0, I_{0,\beta}(\eta(a_\beta)) < 0)\right\}$$

and we have the usual minimax characterization of the ground state

$$\inf_{\eta\in\Gamma_\beta} \max_{t>0} I_{0,\beta}(\eta(t)) = I_{0,\beta}(\widehat{\mathfrak{u}}_\beta, \widehat{\mathfrak{v}}_\beta), \tag{5.4}$$

see e.g. [21, Lemma 3.2].
The next lemma allows us to get the desired lower bound.

Lemma 5.4. *If* $0 < \beta \le 1$*, then*

$$\mathfrak{m}_\beta \ge 2^{\frac{q-2}{q-1}} I_{0,\beta}(\mathfrak{g},0) = 2^{\frac{q-2}{q-1}} \mathcal{I}_{0,0}(\mathfrak{g}).$$

Proof. By Lemma 3.3 and (5.4) it holds

$$\mathfrak{m}_\beta = \max_{t>0} I_{\lambda,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t)) \ge \max_{t>0} I_{0,\beta}(\gamma_{\mathfrak{u}_\beta,\mathfrak{v}_\beta}(t)) \ge \inf_{\eta\in\Gamma_\beta} \max_{t>0} I_{0,\beta}(\eta(t)) = I_{0,\beta}(\widehat{\mathfrak{u}}_\beta, \widehat{\mathfrak{v}}_\beta).$$

On the other hand, by [22, Proof of Lemma 4] we know

$$I_{0,\beta}(\widehat{\mathfrak{u}}_\beta, \widehat{\mathfrak{v}}_\beta) \ge \left(\inf_{k>0} \frac{(1+k^2)^q}{1+k^{2q}+2\beta k^q}\right)^{\frac{1}{q-1}} I_{0,\beta}(\mathfrak{g},0).$$

If we set

$$\xi_\beta(k) := \frac{(1+k^2)^q}{1+k^{2q}+2\beta k^q},$$

we have that, if $\beta_1 < \beta_2$, then, for every $k > 0$, $\xi_{\beta_1}(k) > \xi_{\beta_2}(k)$. Hence, if $\beta \in (0,1]$, it holds

$$\xi_\beta(k) \ge \xi_1(k) \ge \xi_1(1) = 2^{q-2}$$

and the conclusion follows. □

To prove the asymptotic behavior of the ground states $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$, we will show first the asymptotic behavior of the ground state levels. However in order to do that, we will work with another family of ground states different from $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$.

Now we are ready to prove the *asymptotic behavior* as $\beta \to 0^+$ of the ground state levels.

Proposition 5.5. *If* $q \in (3/2, 2)$*, as* $\beta \to 0^+$*, the family of radial ground state solutions of* ($\mathcal{S}_{\lambda,\beta}$) *found in Lemma 3.7 converges in* $\mathrm{H_r}$ *to a semitrivial solution of* ($\mathcal{S}_{\lambda,\beta}$)*, whose nontrivial component is a radial ground state solution of* (1.4)*. Moreover*

$$\lim_{\beta\to 0^+} \mathfrak{m}_\beta = \mathfrak{n}. \tag{5.5}$$

Proof. Using the notations of Lemma 3.7, let us set

$$\mathrm{u}_\beta := \widetilde{\varrho}_\beta \cos\theta_\beta, \quad \mathrm{v}_\beta := \widetilde{\varrho}_\beta \sin\theta_\beta, \quad \widetilde{\varrho}_\beta := \zeta_{\rho_\beta}(t_\beta) = t_\beta^2 \varrho_\beta(t_\beta \cdot).$$

By Remark 3.5 and (5.2) we deduce that

$$(\mathrm{u}_\beta, \mathrm{v}_\beta) \rightharpoonup (\mathrm{u}, \mathrm{v}) \quad \text{in } \mathrm{H_r} \text{ as } \beta \to 0^+.$$

Since $\|(\mathrm{u}_\beta, \mathrm{v}_\beta)\| = \|\widetilde{\varrho}_\beta\|$ and, by Lemma 2.4, $\lim_{\beta\to 0^+} \theta_\beta = \pi/2$, we see that $\mathrm{u}_\beta \to 0$ in $H^1(\mathbb{R}^3)$. Thus $\mathrm{u} = 0$.

We claim that $\mathrm{v} \neq 0$.
By the Sobolev embeddings and the Strauss Lemma, $\mathrm{u}_\beta \to 0$ and $\widetilde{\varrho}_\beta \to \mathrm{v}$ in $L^p(\mathbb{R}^3)$ for all $p \in (2,6)$. Hence, using Lemma 5.4, (3.25), and (ii) in Lemma 2.4,

$$\begin{aligned}0 < 2^{\frac{q-2}{q-1}}\mathcal{I}_{0,0}(\mathfrak{g}) \le I_{\lambda,\beta}(\mathrm{u}_\beta,\mathrm{v}_\beta) &= I_{\lambda,\beta}(\mathrm{u}_\beta,\mathrm{v}_\beta) - \frac{1}{2}I'_{\lambda,\beta}(\mathrm{u}_\beta,\mathrm{v}_\beta)[\mathrm{u}_\beta,\mathrm{v}_\beta]\\ &= -\frac{\lambda}{4}\int \phi_{\widetilde{\varrho}_\beta}\widetilde{\varrho}_\beta^2 + \frac{q-1}{2q}\mathfrak{h}_\beta(y_\beta)\|\widetilde{\varrho}_\beta\|_{2q}^{2q}\\ &\le \frac{q-1}{2q}\mathfrak{h}_\beta(y_\beta)\|\widetilde{\varrho}_\beta\|_{2q}^{2q} \longrightarrow \frac{q-1}{2q}\|\mathrm{v}\|_{2q}^{2q}\end{aligned}$$

getting the claim.
Now we prove that v is a solution of (1.4).
We know that $(\mathrm{u}_\beta,\mathrm{v}_\beta)$ satisfies, for any $\varphi \in H^1_{\mathrm{r}}(\mathbb{R}^3)$,

$$\int \nabla \mathrm{v}_\beta \nabla\varphi + \int \mathrm{v}_\beta\varphi + \lambda\int \phi_{\mathrm{u}_\beta,\mathrm{v}_\beta}\mathrm{v}_\beta\varphi - \int |\mathrm{v}_\beta|^{2\mathrm{q}-2}\mathrm{v}_\beta\varphi - \beta\int |\mathrm{u}_\beta|^{\mathrm{q}}|\mathrm{v}_\beta|^{\mathrm{q}-2}\mathrm{v}_\beta\varphi = 0.$$

Then passing to the limit as $\beta \to 0^+$, using also that, by Lemma 3.2,

$$\left|\int \mathrm{v}_\beta\varphi\phi_{\mathrm{u}_\beta,\mathrm{v}_\beta} - \int \mathrm{v}\varphi\phi_{\mathrm{u},\mathrm{v}}\right| \le \left(\|\phi_{\mathrm{u}_\beta,\mathrm{v}_\beta}\|_6\|\mathrm{v}_\beta-\mathrm{v}\|_{12/5} + \|\phi_{\mathrm{u}_\beta,\mathrm{v}_\beta}-\phi_{\mathrm{u},\mathrm{v}}\|_6\|\mathrm{v}\|_{12/5}\right)\|\varphi\|_{12/5} \to 0,$$

we infer

$$\int \nabla \mathrm{v}\nabla\varphi + \int \mathrm{v}\varphi + \lambda\int \mathrm{v}\varphi\phi_{\mathrm{u},\mathrm{v}} - \int |\mathrm{v}|^{2\mathrm{q}-2}\mathrm{v}\varphi = 0$$

which means that v solves (1.4).
To show the strong convergence $\mathrm{v}_\beta \to \mathrm{v}$ in $H^1_{\mathrm{r}}(\mathbb{R}^3)$, observe that, for all $\psi \in H^1_{\mathrm{r}}(\mathbb{R}^3)$,

$$I'_{\lambda,\beta}(\mathrm{u}_\beta,\mathrm{v}_\beta)[0,\psi] = 0.$$

Then, choosing $\psi = \mathrm{v}_\beta - \mathrm{v}$, we get

$$\begin{aligned}&\int \nabla \mathrm{v}_\beta\nabla(\mathrm{v}_\beta-\mathrm{v}) + \int \mathrm{v}_\beta(\mathrm{v}_\beta-\mathrm{v}) + \lambda\int \mathrm{v}_\beta(\mathrm{v}_\beta-\mathrm{v})\phi_{\mathrm{u}_\beta,\mathrm{v}_\beta}\\ &\qquad = \int |\mathrm{v}_\beta|^{2q-2}\mathrm{v}_\beta(\mathrm{v}_\beta-\mathrm{v}) + \beta\int |\mathrm{u}_\beta|^q|\mathrm{v}_\beta|^{q-2}\mathrm{v}_\beta(\mathrm{v}_\beta-\mathrm{v}).\end{aligned}$$

Passing to the limit as $\beta \to 0^+$ in the above identity, we get $\|\mathrm{v}_\beta\|^2 \to \|\mathrm{v}\|^2$, and so the strong convergence holds.
Finally, using (5.2), we infer

$$\mathfrak{n} > \mathfrak{m}_\beta = I_{\lambda,\beta}(\mathrm{u}_\beta,\mathrm{v}_\beta) \to \mathcal{I}_{\lambda,0}(\mathrm{v}),$$

and then v is a ground state solution of (1.4) and (5.5) follows. □

Hence we can conclude.

Proof of (iii) of Theorem 1.1 (asymptotic behavior). By Remark 3.5 and (5.2) we have that $\{(\mathfrak{u}_\beta, \mathfrak{v}_\beta)\}$ is bounded and then weakly convergent in $\mathrm{H_r}$ to some $(\mathrm{u}^*, \mathrm{v}^*)$.
First we prove that, actually, the convergence is strong.
Indeed, since for every $\psi \in H^1_{\mathrm{r}}(\mathbb{R}^3)$, $I'_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{v}_\beta)[\psi, 0] = 0$, then, choosing $\psi = \mathfrak{u}_\beta - \mathrm{u}^*$ we get, arguing as in the proof of Lemma 5.5, that $\mathfrak{u}_\beta \to \mathrm{u}^*$ in $H^1_{\mathrm{r}}(\mathbb{R}^3)$. Analogously we get $\mathfrak{v}_\beta \to \mathrm{v}^*$ in $H^1_{\mathrm{r}}(\mathbb{R}^3)$ and, arguing as in Lemma 5.5, we see that $(\mathrm{u}^*, \mathrm{v}^*)$ satisfies (4.1).
On the other hand, by (5.5) and the strong convergence of $\{(\mathfrak{u}_\beta, \mathfrak{v}_\beta)\}$ we arrive at

$$I_{\lambda,0}(\mathrm{u}^*, \mathrm{v}^*) = \mathfrak{n} > 0. \tag{5.6}$$

Thus $(\mathrm{u}^*, \mathrm{v}^*) \in \mathrm{H_r} \setminus \{0\}$.
Let us see now that $(\mathrm{u}^*, \mathrm{v}^*)$ is semitrivial.
Using Lemma 4.1 and (5.6), we get

$$\mathfrak{n} = \mathfrak{m}_0 \leq I_{\lambda,0}(\mathrm{u}^*, \mathrm{v}^*) = \mathfrak{n}.$$

Hence, $(\mathrm{u}^*, \mathrm{v}^*)$ is a ground state for (4.1), and so, by item (i) of Theorem 1.1, is semitrivial. □

We conclude this section with a further result about a particular solution of our system.
Let us recall that, as observed in Remark 2.2, $(\mathfrak{z}_\beta, \mathfrak{z}_\beta)$, where $\mathfrak{z}_\beta$ is a ground state solution of (2.5), is a solution of $(\mathcal{S}_{\lambda,\beta})$. Thus, in view of (ii) of Theorem 1.1, for $q \in [2, 3)$, such a solution is not a ground state. The same holds also for $q \in (3/2, 2)$. More precisely we have

Theorem 5.6. *If β is small enough, the couple $(\mathfrak{z}_\beta, \mathfrak{z}_\beta)$ is not a ground state solution of $(\mathcal{S}_{\lambda,\beta})$.*

Let us start with two preliminary lemmata concerning the monotonicity of the ground states levels for a single equation of type (2.5) with respect to the parameters λ and β. Their proofs use standard arguments.

Lemma 5.7. *Let $0 < \lambda_1 < \lambda_2$ and $\mathfrak{w}_i$, $i = 1, 2$ be ground state solutions of*

$$-\Delta u + u + \lambda_i \phi_u u = |u|^{2q-2} u, \ \textit{in}\ \mathbb{R}^3, \quad i = 1, 2.$$

Then

$$\mathcal{I}_{\lambda_1,0}(\mathfrak{w}_1) < \mathcal{I}_{\lambda_2,0}(\mathfrak{w}_2).$$

Proof. Since

$$0 = \mathcal{J}_{\lambda_2,0}(\mathfrak{w}_2) = \mathcal{J}_{\lambda_1,0}(\mathfrak{w}_2) + \frac{3}{4}(\lambda_2 - \lambda_1) \int \mathfrak{w}_2^2 \phi_{\mathfrak{w}_2} > \mathcal{J}_{\lambda_1,0}(\mathfrak{w}_2) = \mathcal{J}_{\lambda_1,0}(\zeta_{\mathfrak{w}_2}(1))$$

and by (2.3) and (a) in Lemma 2.3, we see that there exists $t_1 \in (0, 1)$ such that

$$\mathcal{J}_{\lambda_1,0}(\zeta_{\mathfrak{w}_2}(t_1)) = 0,$$

namely, $\zeta_{\mathfrak{w}_2}(t_1) \in \mathcal{N}^{\lambda_1}$.
Therefore

$$\mathcal{I}_{\lambda_1,0}(\mathfrak{w}_1) \le \mathcal{I}_{\lambda_1,0}(\zeta_{\mathfrak{w}_2}(t_1)) < \mathcal{I}_{\lambda_2,0}(\zeta_{\mathfrak{w}_2}(t_1)) < \mathcal{I}_{\lambda_2,0}(\zeta_{\mathfrak{w}_2}(1)) = \mathcal{I}_{\lambda_2,0}(\mathfrak{w}_2),$$

concluding the proof. □

Lemma 5.8. *For* $0 \le \beta_1 < \beta_2$*, let* $\mathfrak{z}_{\beta_1}, \mathfrak{z}_{\beta_2}$ *be respective ground states of* (2.5)*. Then*

$$0 < \mathcal{I}_{2\lambda,\beta_2}(\mathfrak{z}_{\beta_2}) < \mathcal{I}_{2\lambda,\beta_1}(\mathfrak{z}_{\beta_1}).$$

Proof. We know that

$$\begin{aligned}
0 &= \mathcal{J}_{2\lambda,\beta_1}(\mathfrak{z}_{\beta_1}) \\
&= \frac{3}{2}\|\nabla \mathfrak{z}_{\beta_1}\|_2^2 + \frac{1}{2}\|\mathfrak{z}_{\beta_1}\|_2^2 + \frac{3}{2}\lambda \int \phi_{\mathfrak{z}_{\beta_1}} \mathfrak{z}_{\beta_1}^2 - \frac{4q-3}{2q}(1+\beta_1)\|\mathfrak{z}_{\beta_1}\|_{2q}^{2q} \\
&> \frac{3}{2}\|\nabla \mathfrak{z}_{\beta_1}\|_2^2 + \frac{1}{2}\|\mathfrak{z}_{\beta_1}\|_2^2 + \frac{3}{2}\lambda \int \phi_{\mathfrak{z}_{\beta_1}} \mathfrak{z}_{\beta_1}^2 - \frac{4q-3}{2q}(1+\beta_2)\|\mathfrak{z}_{\beta_1}\|_{2q}^{2q} \\
&= \mathcal{J}_{2\lambda,\beta_2}(\mathfrak{z}_{\beta_1}).
\end{aligned}$$

Hence, by (2.3) and (a) in Lemma 2.3, there exists $t_{\beta_1} \in (0,1)$ such that

$$\mathcal{J}_{2\lambda,\beta_2}(\zeta_{\mathfrak{z}_{\beta_1}}(t_{\beta_1})) = 0.$$

Then,

$$0 < \mathcal{I}_{2\lambda,\beta_2}(\mathfrak{z}_{\beta_2}) \le \mathcal{I}_{2\lambda,\beta_2}(\zeta_{\mathfrak{z}_{\beta_1}}(t_{\beta_1})) < \mathcal{I}_{2\lambda,\beta_1}(\zeta_{\mathfrak{z}_{\beta_1}}(t_{\beta_1})) < \mathcal{I}_{2\lambda,\beta_1}(\zeta_{\mathfrak{z}_{\beta_1}}(1)) = \mathcal{I}_{2\lambda,\beta_1}(\mathfrak{z}_{\beta_1})$$

and the proof is complete. □

In particular Lemma 5.8 says that, if $\beta > 0$,

$$\mathcal{I}_{2\lambda,\beta}(\mathfrak{z}_\beta) < \mathcal{I}_{2\lambda,0}(\mathfrak{z}_0). \tag{5.7}$$

Then we can prove the desired result.

Proof of Theorem 5.6. By Lemma 5.7, with $\lambda_1 = \lambda, \lambda_2 = 2\lambda, \mathfrak{w}_1 = \mathfrak{w}, \mathfrak{w}_2 = \mathfrak{z}_0$, and (2.8), we deduce

$$\mathfrak{n} = \mathcal{I}_{\lambda,0}(\mathfrak{w}) < \mathcal{I}_{2\lambda,0}(\mathfrak{z}_0) < 2\mathcal{I}_{2\lambda,0}(\mathfrak{z}_0) = I_{\lambda,0}(\mathfrak{z}_0, \mathfrak{z}_0). \tag{5.8}$$

Let now $\{\beta_n\} \subset (0, +\infty)$ be such that $\beta_n \to 0^+$ and $\beta_{n+1} < \beta_n$ and $k_{\beta_n} := \mathcal{I}_{2\lambda,\beta_n}(\mathfrak{z}_{\beta_n}) > 0$. By Lemma 5.8 we know that $\{k_{\beta_n}\}$ is bounded and, by (5.7),

$$0 < k_{\beta_0} < k_{\beta_n} < \mathcal{I}_{2\lambda,0}(\mathfrak{z}_0). \tag{5.9}$$

Arguing for the single equation (2.5) as in (3.13) and Remark 3.5, we get that $\{\mathfrak{z}_{\beta_n}\}$ is bounded in $H^1_{\rm r}(\mathbb{R}^3)$ and we know also that $\mathcal{I}'_{2\lambda,\beta_n}(\mathfrak{z}_{\beta_n})=0$. Thus

$$\mathcal{I}_{2\lambda,0}(\mathfrak{z}_{\beta_n}) = k_{\beta_n} + \frac{\beta_n}{2q}\|\mathfrak{z}_{\beta_n}\|_{2q}^{2q} = k_{\beta_n} + \varepsilon_n$$

and

$$\|\mathcal{I}'_{2\lambda,0}(\mathfrak{z}_{\beta_n})\| = \sup_{\|v\|\le 1}\left|\mathcal{I}'_{2\lambda,\beta_n}(\mathfrak{z}_{\beta_n})[v] + \beta_n\int |\mathfrak{z}_{\beta_n}|^{2q-2}\mathfrak{z}_{\beta_n} v\right| \le C\beta_n\|\mathfrak{z}_{\beta_n}\|^{2q-1} = \varepsilon_n,$$

namely that $\{\mathfrak{z}_{\beta_n}\}$ is a (PS) sequence for $\mathcal{I}_{2\lambda,0}$.
Arguing as in the Proof of Proposition 5.5, we show that $\mathfrak{z}_{\beta_n}\to w$ in $H^1_{\rm r}(\mathbb{R}^3)$, $\mathcal{I}'_{2\lambda,0}(w)=0$, and, by (5.9), $w\neq 0$.
Moreover,

$$k_{\beta_n} = \mathcal{I}_{2\lambda,0}(\mathfrak{z}_{\beta_n}) - \frac{\beta_n}{2q}\|\mathfrak{z}_{\beta_n}\|_{2q}^{2q} \to \mathcal{I}_{2\lambda,0}(w).$$

Hence, by (2.8) and (5.8),

$$I_{\lambda,\beta_n}(\mathfrak{z}_{\beta_n},\mathfrak{z}_{\beta_n}) = 2k_{\beta_n} \to 2\mathcal{I}_{2\lambda,0}(w) \ge 2\mathcal{I}_{2\lambda,0}(\mathfrak{z}_0) = I_{\lambda,0}(\mathfrak{z}_0,\mathfrak{z}_0) > \mathfrak{n}.$$

Thus, by Proposition 5.5, we get that for β small $(\mathfrak{z}_\beta,\mathfrak{z}_\beta)$ cannot be a ground state. □

6. The case β large

In this section we study the *vectorial nature* of the ground states $(\mathfrak{u}_\beta,\mathfrak{v}_\beta)$ of $(\mathcal{S}_{\lambda,\beta})$ for β large, namely satisfying (1.5), and we show that such a ground state vanishes as $\beta\to+\infty$. Indeed we have

Proof of (iv) of Theorem 1.1. The fact that the ground state solution has to be vectorial, follows taking into account that, by (ii) and (c) of (iii) in Lemma 2.4, $\max_{[0,1]}\mathfrak{h}_\beta > 1$. Then arguing as in Lemma 5.2, this implies that $\mathfrak{n} > \mathfrak{m}_\beta$ and so we can conclude as in the proof of (ii) of Theorem 1.1.
To prove (1.6), let us fix $u\in H^1_{\rm r}(\mathbb{R}^3)\setminus\{0\}$ and let $T_\beta>0$ be the real number such that $J_{\lambda,\beta}(\gamma_{u,u}(T_\beta))=0$, namely such that $\gamma_{u,u}(T_\beta)\in\mathcal{M}$. By (c) in Lemma 2.3 we have that

$$\lim_{\beta\to+\infty} T_\beta = 0 \text{ and } \lim_{\beta\to+\infty}\frac{4q-3}{q}(1+\beta)\|u\|_{2q}^{2q}T_\beta^{4q-4} = \|u\|_2^2.$$

Thus

$$\begin{aligned}0 &< \mathfrak{m}_\beta \le I_{\lambda,\beta}(\gamma_{u,u}(T_\beta))\\ &= T_\beta^3\|\nabla u\|_2^2 + T_\beta\|u\|_2^2 + \lambda T_\beta^3\int \phi_u u^2 - \frac{T_\beta^{4q-3}}{q}(1+\beta)\|u\|_{2q}^{2q}\end{aligned}$$

$$= T_\beta \|u\|_2^2 + \frac{1}{3}\left[\frac{4q-3}{q}(1+\beta)T_\beta^{4q-3}\|u\|_{2q}^{2q} - T_\beta\|u\|_2^2\right] - \frac{T_\beta^{4q-3}}{q}(1+\beta)\|u\|_{2q}^{2q}$$
$$= \frac{2}{3}T_\beta\left(\|u\|_2^2 - \frac{2q-3}{q}(1+\beta)T_\beta^{4q-4}\|u\|_{2q}^{2q}\right) \to 0$$

as $\beta \to +\infty$.
Hence we conclude by Remark 3.5. □

Now we show that, actually, a ground state can be taken with the two components equal. Indeed

Theorem 6.1. *If β satisfies (1.5), then*

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\mathfrak{z}_\beta, \mathfrak{z}_\beta),$$

where $\mathfrak{z}_\beta$ is a ground state solution of (2.5).

Let $(\mathfrak{u}_\beta, \mathfrak{v}_\beta)$ be a vectorial ground state just found and let us consider its polar coordinates as in (3.24). Taking into account Lemma 3.7 and so Remark 3.8, using (ii) and (c) of (iii) in Lemma 2.4, we have

Lemma 6.2. *If β satisfies (1.5), then there exists $t_\beta > 0$ such that $\gamma_{\varrho_\beta/\sqrt{2},\varrho_\beta/\sqrt{2}}(t_\beta) \in \mathcal{M}$ and*

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\gamma_{\varrho_\beta/\sqrt{2},\varrho_\beta/\sqrt{2}}(t_\beta)).$$

In particular $\gamma_{\varrho_\beta/\sqrt{2},\varrho_\beta/\sqrt{2}}(t_\beta)$ is a ground state solution.

Thus we are ready to complete the proof.

Proof of Theorem 6.1. By Lemma 6.2, there exists $\mathfrak{u}_\beta \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\mathfrak{u}_\beta, \mathfrak{u}_\beta).$$

Thus, by Remark 2.2, we infer

$$\mathfrak{m}_\beta = 2\mathcal{I}_{2\lambda,\beta}(\mathfrak{u}_\beta) \geq 2\mathcal{I}_{2\lambda,\beta}(\mathfrak{z}_\beta) = I_{\lambda,\beta}(\mathfrak{z}_\beta, \mathfrak{z}_\beta) \geq \mathfrak{m}_\beta$$

concluding the proof. □

7. The particular case $\beta = 2^{q-1} - 1$ and $q \in [2, 3)$

In this particular case we can argue as in Section 5 and Section 6 to get both (semitrivial and vectorial) types of ground states.
By Lemma 5.1, being $\mathfrak{m}_\beta = \mathfrak{n}$, we get that

$$I_{\lambda,\beta}(\mathfrak{w}, 0) = I_{\lambda,\beta}(0, \mathfrak{w}) = \mathfrak{m}_\beta$$

and so $(\mathfrak{w}, 0)$ and $(0, \mathfrak{w})$ are semitrivial ground states.
Of course in this case we cannot proceed as in the proof of item (i) of Theorem 1.1 since 0 and 1 are not the only maximisers of $\mathfrak{h}_\beta$ and so, the existence of further maximisers gives vectorial ground states too (see (b) of item (iii) in Lemma 2.4).
Indeed Lemma 3.7 applies with $y_\beta = 1/2$, and so $\theta_\beta = \pi/4$. Hence we get that there exists $t_\beta > 0$ such that $\gamma_{\varrho_\beta/\sqrt{2},\varrho_\beta/\sqrt{2}}(t_\beta) \in \mathcal{M}$,

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\gamma_{\varrho_\beta/\sqrt{2},\varrho_\beta/\sqrt{2}}(t_\beta)),$$

and $\gamma_{\varrho_\beta/\sqrt{2},\varrho_\beta/\sqrt{2}}(t_\beta)$ is a ground state solution.
More in particular, if $q = 2$, $\mathfrak{h}_\beta \equiv 1$. Then we can take an arbitrary $y_\beta \in (0, 1)$, obtaining that there exists $t_\beta > 0$ such that $\gamma_{\varrho_\beta \cos\theta_\beta,\varrho_\beta \sin\theta_\beta}(t_\beta)$ is a ground state solution.
Finally we observe that, as a corollary of this last property, arguing as in Theorem 6.1, we have also that

$$\mathfrak{m}_\beta = I_{\lambda,\beta}(\mathfrak{z}_\beta, \mathfrak{z}_\beta).$$

Data availability

No data was used for the research described in the article.

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Appendix A. Proof of Lemma 2.4

In this Appendix we present the details of the proof of Lemma 2.4.
First observe that $\mathfrak{h}_\beta$ is even with respect to the line $y = 1/2$.
Since $\mathfrak{h}_0$ is strictly decreasing in $[0, 1/2]$, property (i) is trivial.
Now let us consider $\beta > 0$. The proof when $q = 2$ is trivial. Thus, let us focus on (iii) for $q \in (2, 3)$ and (ii).
Observe that, for any fixed $\beta > 0$, we have that $\mathfrak{h}'_\beta(1/2) = 0$ and $\mathfrak{h}''_\beta(1/2) = 2^{3-q} q(q - 1 - \beta)$.
Thus, if $\beta < q - 1$, then, $1/2$ cannot be a maximum point of $\mathfrak{h}_\beta$.
Moreover, in $(0, 1/2]$,

$$\frac{\mathfrak{h}'_\beta(y)}{q(1-y)^{q-1}} = \left(\frac{y}{1-y}\right)^{q-1} - 1 + \beta\left(\frac{y}{1-y}\right)^{\frac{q}{2}-1} - \beta\left(\frac{y}{1-y}\right)^{\frac{q}{2}} \tag{A.1}$$

Thus, to study the sign of $\mathfrak{h}'_\beta$, we can consider the right hand side of (A.1) and, for simplicity, we write it as

$$\mathfrak{g}_\beta(t) := t^{q-1} - 1 + \beta t^{q/2-1} - \beta t^{q/2}, \quad t \in (0,1],$$

whose derivative is

$$\mathfrak{g}'_\beta(t) = \frac{\mathfrak{r}_\beta(t)}{2t^{2-q/2}} \quad \text{where } \mathfrak{r}_\beta(t) := 2(q-1)t^{q/2} - \beta q t + \beta(q-2).$$

Let us prove (ii).
Note that, whenever $q \in (3/2, 2)$ and $\beta > 0$, we have that $\lim_{y\to 0^+} \mathfrak{h}'_\beta(y) = +\infty$. Thus, since $\mathfrak{h}_\beta(0) = 1$, we get the existence of $y_\beta \in (0, 1/2]$ such that $\mathfrak{h}_\beta(y_\beta) = \max_{y\in[0,1]} \mathfrak{h}_\beta(y) > 1$.
Moreover

$$\lim_{t\to 0^+} \mathfrak{g}_\beta(t) = +\infty \text{ and } \mathfrak{g}_\beta(1) = 0. \tag{A.2}$$

If $\beta \in (0, q-1)$ we have that $\mathfrak{r}_\beta(0) < 0$, $\mathfrak{r}_\beta(1) > 0$, and $\mathfrak{r}_\beta$ is (strictly) increasing on the left of its unique maximum point $((q-1)/\beta)^{2/(2-q)}$ and then it is (strictly) decreasing. Thus $\mathfrak{r}_\beta$ has a unique zero $\mathfrak{t}_\beta$ which is the unique critical point (minimizer) of $\mathfrak{g}_\beta$ and $\mathfrak{g}_\beta$ is (strictly) decreasing in $(0, \mathfrak{t}_\beta)$ and (strictly) increasing in $(\mathfrak{t}_\beta, 1)$. Hence, by (A.2), $\mathfrak{g}_\beta$ has a unique zero in $(0, 1)$ which gives us the unique maximum point y_β.
If $\beta = q-1$ we have that $\mathfrak{r}_\beta(0) < 0$, $\mathfrak{r}_\beta(1) = 0$, and $\mathfrak{r}_\beta$ is (strictly) increasing in $(0, 1)$. Thus $\mathfrak{g}'_\beta$ is (strictly) negative in $(0, 1)$ and so, by (A.1) and (A.2), $\mathfrak{h}'_\beta$ is (strictly) positive in $(0, 1/2)$. Hence the symmetry of $\mathfrak{h}_\beta$ allows us to conclude.
If $\beta > q-1$ we have that $\mathfrak{r}_\beta(0) < 0$, $\mathfrak{r}_\beta(1) < 0$, and $\mathfrak{r}_\beta$ is (strictly) increasing on the left of its unique maximum point $((q-1)/\beta)^{2/(2-q)}$ and then it is (strictly) decreasing. Moreover

$$\mathfrak{r}_\beta\left(((q-1)/\beta)^{2/(2-q)}\right) = \frac{2-q}{\beta^{q/(2-q)}}\left((q-1)^{2/(2-q)} - \beta^{2/(2-q)}\right) < 0.$$

Then, $\mathfrak{r}_\beta$ is (strictly) negative in $(0, 1)$ and so, by (A.1) and (A.2), $\mathfrak{h}'_\beta$ is (strictly) positive in $(0, 1/2)$. Hence we can conclude as in the previous step.
To prove the asymptotic behavior of y_β as $\beta \to 0^+$, let us recall that $y_\beta \in (0, 1/2)$ for $\beta < q-1$. If, by contradiction, we assume that $y_\beta \not\to 0$ as $\beta \to 0^+$, then there exists a sequence $\{\beta_n\}$ tending to zero, such that $\mathfrak{h}_{\beta_n}(y_{\beta_n}) > 1$ and $\lim_n y_{\beta_n} = \ell \in (0, 1/2]$. Then

$$1 \le \lim_n \mathfrak{h}_{\beta_n}(y_{\beta_n}) = \ell^q + (1-\ell)^q \le \frac{1}{2^{q-1}} < 1$$

getting the contradiction.
Let us prove (iii).
In this case, namely whenever $q \in (2, 3)$ and $\beta > 0$, we have that $\mathfrak{h}'_\beta(0) = -q$. Moreover

$$\mathfrak{g}_\beta(0) = -1 \text{ and } \mathfrak{g}_\beta(1) = 0. \tag{A.3}$$

If $\beta \in (0, q-1)$, we have that $\mathfrak{r}_\beta(0) > 0$, $\mathfrak{r}_\beta(1) > 0$, and $\mathfrak{r}_\beta$ is (strictly) decreasing before its unique minimum point $(\beta/(q-1))^{2/(q-2)}$ and then it is (strictly) increasing. Moreover

$$\mathfrak{r}_\beta\Big((\beta/(q-1))^{2/(q-2)}\Big) = \frac{\beta(q-2)}{(q-1)^{2/(q-2)}}\big((q-1)^{2/(q-2)} - \beta^{2/(q-2)}\big) > 0.$$

Thus, $\mathfrak{r}_\beta$ is (strictly) positive in $(0, 1)$ and so, by (A.1) and (A.3), $\mathfrak{h}'_\beta$ is (strictly) negative in $(0, 1/2)$. Hence the symmetry of $\mathfrak{h}_\beta$ allows us to conclude.

If $\beta = q-1$ we have that $\mathfrak{r}_\beta(0) > 0$, $\mathfrak{r}_\beta(1) = 0$, and $\mathfrak{r}_\beta$ is (strictly) decreasing in $(0, 1)$. Thus $\mathfrak{g}'_\beta$ is (strictly) positive in $(0, 1)$ and so, by (A.1) and (A.3), $\mathfrak{h}'_\beta$ is (strictly) negative in $(0, 1/2)$. Hence we can conclude as in the previous step.

If $\beta > q-1$, we have that $\mathfrak{r}_\beta(0) > 0$, $\mathfrak{r}_\beta(1) < 0$, and $\mathfrak{r}_\beta$ is (strictly) decreasing in $(0, 1)$. Thus $\mathfrak{r}_\beta$ has a unique zero $\mathfrak{t}_\beta$ which is the unique critical point (maximum point) of $\mathfrak{g}_\beta$ and $\mathfrak{g}_\beta$ is (strictly) increasing in $(0, \mathfrak{t}_\beta)$ and (strictly) decreasing in $(\mathfrak{t}_\beta, 1)$. Hence, by (A.3), $\mathfrak{g}_\beta$ has a unique zero in $(0, 1)$ which gives us a unique minimum point of $\mathfrak{h}_\beta$ in $(0, 1/2)$ and so, the unique local maximum point of $\mathfrak{h}_\beta$ in $(0, 1)$ is $1/2$.

Since $\mathfrak{h}_\beta(0) = \mathfrak{h}_\beta(1) = 1$ and $\mathfrak{h}_\beta(1/2) = (1+\beta)/2^{q-1}$ we get that $1/2$ is the global maximum point of $\mathfrak{h}_\beta$ in $[0, 1/2]$ if and only if $\beta \geq 2^{q-1} - 1$ and it is the unique global maximum if and only if $\beta > 2^{q-1} - 1$.

References

[1] A. Ambrosetti, E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc. 75 (2007) 67–82.

[2] T. Bartsch, Z.-Q. Wang, Note on ground states of nonlinear Schrödinger systems, J. Partial Differ. Equ. 19 (2006) 200–207.

[3] H. Berestycki, P.L. Lions, Nonlinear scalar fields equations I, existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983) 313–345.

[4] O. Bokanowski, J.L. López, J. Soler, On an exchange interaction model for quantum transport: the Schrödinger-Poisson-Slater system, Math. Models Methods Appl. Sci. 13 (2003) 1397–1412.

[5] J.M. Combes, P. Duclos, R. Seiler, The Born-Oppenheimer approximation, in: G. Velo, A.S. Wightman (Eds.), Rigorous Atomic and Molecular Physics, in: NATO Advanced Study Institutes Series, vol. 74, Springer, Boston, MA, 1981.

[6] M. Conti, S. Terracini, G. Verzini, Nehari's problem and competing species systems, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 19 (2002) 871–888.

[7] M. Conti, S. Terracini, G. Verzini, Asymptotic estimates for the spatial segregation of competitive systems, Adv. Math. 195 (2005) 524–560.

[8] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, Adv. Nonlinear Stud. 4 (2004) 307–322.

[9] M. Defranceschi, C. Le Bris, Mathematical Models and Methods for Ab Initio Quantum Chemistry, Lecture Notes in Chem., vol. 74, Springer, Berlin, 2000.

[10] P.A.M. Dirac, Note on exchange phenomena in the Thomas atom, Proc. Camb. Philos. Soc. 26 (1931) 376–385.

[11] A. Faraj, A. Mantile, F. Nier, Double scale analysis of a Schrödinger-Poisson system with quantum wells and macroscopic nonlinearities in dimension 2 and 3, Asymptot. Anal. 62 (2009) 163–205.

[12] V. Fock, Näherungsmethode zur Lösung des quantenmechanischen Mehrkörperproblems, Z. Phys. 61 (1930) 126–148.

[13] J. Fröhlich, A. Knowles, A microscopic derivation of the time-dependent Hartree-Fock equation with Coulomb two-body interaction, J. Stat. Phys. 145 (2011) 23–50.

[14] R. Gaspar, Über eine approximation des Hartree-Fockschen potentials durch eine universelle potentialfunktion, Acta Phys. Hung. 3 (1954) 263–285.

[15] D. Hartree, The wave mechanics of an atom with a non-Coulomb central field. Part I. Theory and methods, Proc. Camb. Philol. Soc. 24 (1928) 89–312.
[16] W. Kohn, L.J. Sham, Self-consistent equations including exchange and correlation effects, Phys. Rev. 140 (1965) A1133–A1138.
[17] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1977) 93–105.
[18] E.H. Lieb, B. Simon, The Hartree-Fock theory for Coulomb systems, Commun. Math. Phys. 53 (1977) 185–194.
[19] T.-C. Lin, J. Wei, Ground state of n coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, Commun. Math. Phys. 255 (2005) 629–653.
[20] P.-L. Lions, Solutions of Hartree-Fock equations for Coulomb systems, Commun. Math. Phys. 109 (1987) 33–97.
[21] L.A. Maia, E. Montefusco, B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, J. Differ. Equ. 229 (2006) 743–767.
[22] R. Mandel, Minimal energy solutions for cooperative nonlinear Schrödinger systems, Nonlinear Differ. Equ. Appl. 22 (2015) 239–262.
[23] N.J. Mauser, The Schrödinger-Poisson-X_α equation, Appl. Math. Lett. 14 (2001) 759–763.
[24] R. McWeeny, Methods of Molecular Quantum Mechanics, Academic Press, San Diego, 1992.
[25] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013) 153–184.
[26] R.S. Palais, The principle of symmetric criticality, Commun. Math. Phys. 69 (1979) 19–30.
[27] R.G. Parr, W. Yang, Density-Functional Theory of Atoms and Molecules, Oxford Univ. Press, New York, 1989.
[28] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237 (2006) 655–674.
[29] B. Sirakov, Least energy solitary waves for a system of nonlinear Schroödinger equations in $\mathbb{R}^n$, Commun. Math. Phys. 271 (2007) 199–221.
[30] J.C. Slater, A note on Hartree's method, Phys. Rev. 35 (1930) 210–211.
[31] J.C. Slater, A simplification of the Hartree-Fock method, Phys. Rev. 81 (1951) 385–390.
[32] N. Soave, On existence and phase separation of solitary waves for nonlinear Schrödinger systems modelling simultaneous cooperation and competition, Calc. Var. Partial Differ. Equ. 53 (2015) 689–718.
[33] W.A. Strauss, Existence of solitary waves in higher dimensions, Commun. Math. Phys. 55 (1977) 149–162.
[34] A. Szabo, N.S. Ostlund, Modern Quantum Chemistry: Introduction to Advanced Electronic Structure Theory, Dover Publ., Mineola, New York, 1996.
[35] J. Wei, M. Winter, Strongly interacting bumps for the Schrödinger-Newton equations, J. Math. Phys. 50 (2009) 012905.
[36] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations Appl., vol. 24, Birkhäuser, Boston, 1996.